Causal Inference Methods in Data Science Lecture 4: Advanced Graphical Models: MEC, CPDAG, ADMG, Causal Discovery and Tian's ID algorithm

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July 4, 2022

Preface

Most causal inference researchers in statistics do not understand graphical models to the level of doing creative research in this field but obviously people started to realize the importance of better combining causal graphs with statistical inference around about 2019

After this lecture, you could read the following people's recent papers if want to work on this field (without particular ordering):
Ilya Shpitser, Elias Barenboim, Thomas Richardson, Robin Evans, F. Richard Guo, Emilija Perković, Marloes Maathius, Yangbo He, Jiji Zhang, Jin Tian, Steffan Lauritzen, Samuel Wang, Caroline Uhler

Why causal graphs?

- As we have seen, causal inference relies heavily on background knowledge
- Causal graph is a very succinct way of representing background knowledge
- We have seen some examples: backdoor, frontdoor
- But ...

Motivating problems

- In certain applications, we do not have much background knowledge to begin with. But it might be easy to collect data (e.g. in biology). Can we learn background knowledge in the form of causal graph from data?

- If the causal graph is very complicated (like below), how can you identify any given causal query (in terms of counterfactuals)?

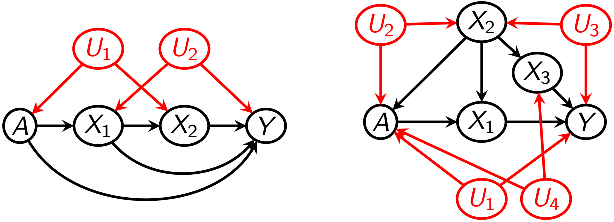


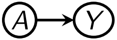
Figure: Is $p(Y(a))$ identifiable in the above two graphs?

- Finally, in complex causal graphs, the same causal query might have different identified formula. Which one should we use?

Causal discovery or structure learning: From data to causal graphs

Markov equivalence class (MEC) of a causal DAG

- Suppose the underlying but unknown causal DAG $\mathcal{G} = (V, E)$ is



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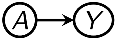

- Qn: if we only observe $(A_i, Y_i)_{i=1}^n$, can we recover the above graph? Or equivalently, can we distinguish the following three structures? (1) $A \rightarrow Y$; (2) $A \leftarrow Y$; (3) $A \quad\quad Y$

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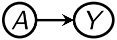

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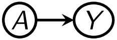

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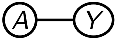
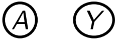
- Without modeling assumptions, one cannot infer the direction so the discovered graph is one of the following MECs (represented by PDAGs) of DAGs between A and Y:





Countering MEC?

The conclusion that we cannot distinguish between $A \longrightarrow Y$ and $A \longleftarrow Y$ is a “nonparametric” statement, in the following sense:

For any distribution P Markov factorized according to $A \longrightarrow Y$, one can always find a distribution Q Markov factorized according to $A \longleftarrow Y$ such that $P \sim Q$

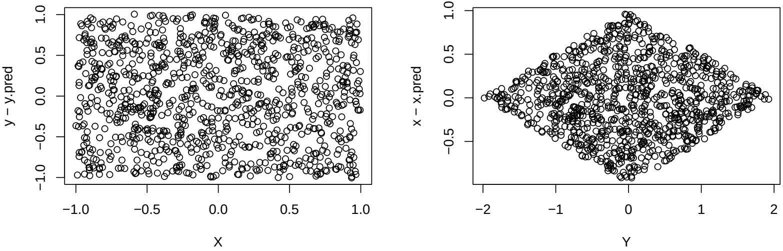
But it does not rule out the possibility of distinguishing DAGs in one MEC by imposing more modeling assumptions: e.g. linear non-Gaussian models

Distinguishable orientation based on Linear non-Gaussian models

Let's assume the following SEM:

$$A \sim \text{Unif}([-1,1]), Y \sim A + \text{Unif}([-1,1])$$

We could fit two linear regressions $\text{lm}(Y \sim A - 1)$ and $\text{lm}(A \sim Y - 1)$ and look at their residual plots:



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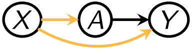

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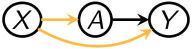

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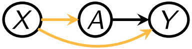

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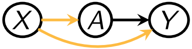

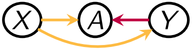

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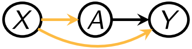

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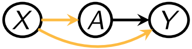
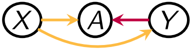
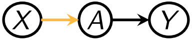




2) No other independence/conditional independence tests can further orient the uncertainty

Example 2



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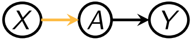

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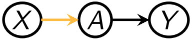

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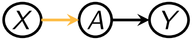

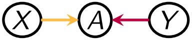

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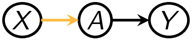

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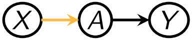

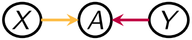

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But impossibility results by Jonas Peters and Rajen Shah AoS 2020 and Neykov, Balakrishnan, Wasserman AoS 2022 when the variable being conditioned on is continuous

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Markov equivalence class (MEC) of a causal DAG $\mathcal{G}_0$: a set of causal DAGs $[\mathcal{G}]$ containing $\mathcal{G}_0$ such that one cannot distinguish among members in $[\mathcal{G}]$ with only observational data

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WARNING: v-structure means $V_1 \rightarrow V_3 \leftarrow V_2$ and there shall be no arrows between V_1 and V_2; v-structure is also called "unshielded collider"

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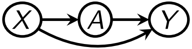
MEC of the following graph $\mathcal{G}_0$ (note: no v-structure)



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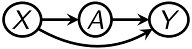

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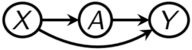

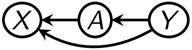

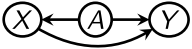

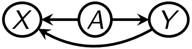

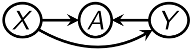

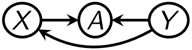

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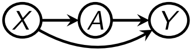
  2) If $X \rightarrow Y$ is in every DAG in $[\mathcal{G}]$, then $X \rightarrow Y$ in $\mathcal{C}$

Example

CPDAG of



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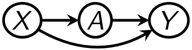

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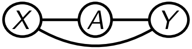

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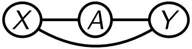
  2) Background knowledge or doing intervention

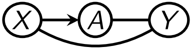
From CPDAG to MPDAG

The power of background knowledge or simply doing intervention (action in reinforcement learning):

Example:



if we additionally have background knowledge $X \to A$, then we have the following Maximal PDAG (MPDAG), denoted as $\mathcal{M}$



how about we intervene X? can we have more precise MPDAG?

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3) Given identifiable causal query, what is the optimal identification formula? (Rotnitzky and Smucler, JMLR 2020)

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1) Given purely observational data, what algorithm can recover the CPDAG? (PC algorithm)

2) Given a DAG/CPDAG, what causal query can be identified? (see Elias Bareinboim's recent papers)

3) Given identifiable causal query, what is the optimal identification formula? (Rotnitzky and Smucler, JMLR 2020)

4) 1), 2), 3) with interventional data (or background knowledge) and/or allowing for latent factors

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- Ground rule 2: if $X \quad Y$ but $X \rightarrow Z \leftarrow Y$ (v-structure), then

$$Z \notin \boldsymbol{S}_{X,Y}$$

PC algorithm: vanilla version

Initialize a complete undirected graph

1) For every pair (X, Y), if there exists $\boldsymbol{S}_{X,Y} \subseteq V \setminus \{X, Y\}$ such that

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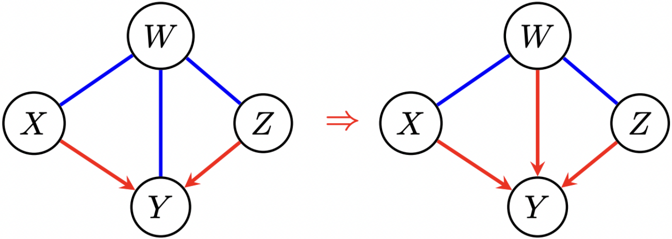
3) Apply the following sub-rules:
    i) $X \quad Z$ and $X \rightarrow Y - Z \Rightarrow X \rightarrow Y \rightarrow Z$ [due to 2) and logic]
    ii) $X \rightarrow Y \rightarrow Z$ and $X - Z \Rightarrow X \rightarrow Z$ [acyclicity]
    iii) $X \quad Z$, $X - W - Z$, $X \rightarrow Y \leftarrow Z$ and $W - Y \Rightarrow W \rightarrow Y$ [why?]



Why not $Y \rightarrow W$? If it were the case, then also need to orient $X \rightarrow W$ and $Z \rightarrow W$ by sub-rule 3):ii) so we have $X \rightarrow W \leftarrow Z$ and $X \quad Z$

But we should have had oriented such v-structure in step 2)

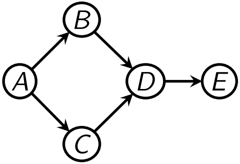
PC algorithm: improve computational efficiency

- Intractable: for each pair (X, Y), if there are p vertices in the graph, then to find $\boldsymbol{S}_{X,Y}$ one needs to enumerate over 2^{p-2} possible sets of vertices

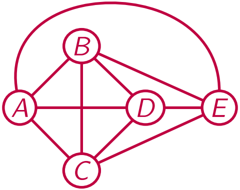
PC algorithm: improve computational efficiency

- Intractable: for each pair (X, Y), if there are p vertices in the graph, then to find $\boldsymbol{S}_{X,Y}$ one needs to enumerate over 2^{p-2} possible sets of vertices
- See the following paper on how to improve computational efficiency
  REF: Computation, causation and discovery
  Main idea: searching for $\boldsymbol{S}_{X,Y}$, starting from empty set, increasing the cardinality one by one

Example



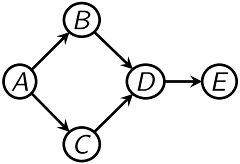
Unknown true graph



Initial complete undirected graph

Check marginal independencies

No pairs are independent marginally. Nothing changed



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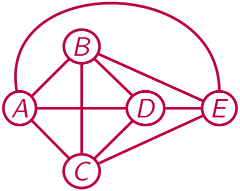
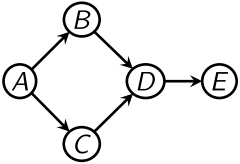


Initial complete undirected graph

Check conditional independencies

$B \perp\!\!\!\perp C | A$



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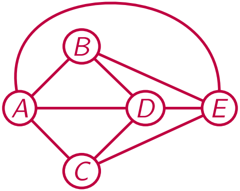
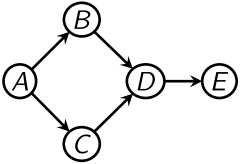


Check conditional independencies

$A \perp\!\!\!\perp E | D$



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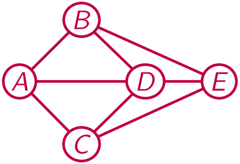
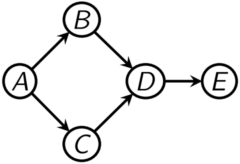


Check conditional independencies

$B \perp\!\!\!\perp E|D$, $C \perp\!\!\!\perp E|D$



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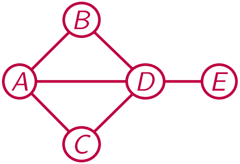
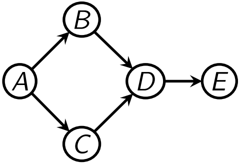


Check conditional independencies

$A \perp\!\!\!\perp D | \{B, C\}$



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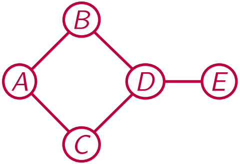
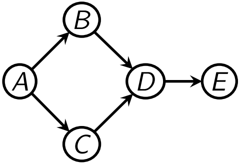


Skeleton of the unknown true graph

Orientation of edges in the skeleton

Rule 2) (about v-structure)



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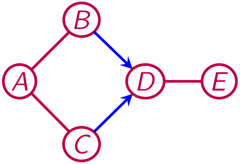

Orientation of the v-structure

Orientation of edges in the skeleton

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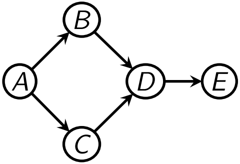

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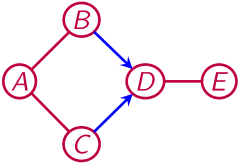

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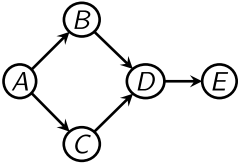
Rule 3):i)



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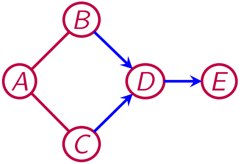

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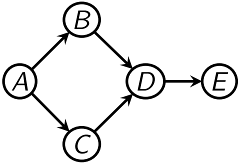

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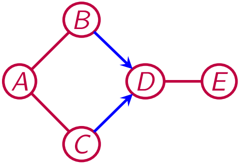
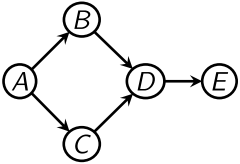


Orientation of the v-structure

Rule 3):i)



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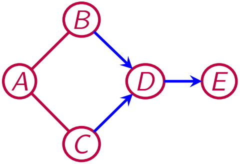

Do we need to consider Rule 3):iii)?

Theoretical guarantee

Spirtes and Glymour showed consistency of PC algorithm:

Theorem 1

Under the faithfulness assumption, with input data $V_i, i = 1, \cdots, n$, the output of the PC algorithm $\widehat{\mathcal{C}}$ converges to the true CPDAG $\mathcal{C}$ of the underlying unknown DAG $\mathcal{G}$, as $n \to \infty$.

Other algorithms

- As always, there are almost no problems that can only be solved by one algorithm

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- Maathuis et al. Nature Methods 2009: IDA algorithm (but assuming linear models so bummer)
  By far, the most influential work in causal discovery in applications (genomics).
  But in genomics, whether it provides significant gain compared to existing correlation/partial correlation based method still remains to be seen.

More recent works

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- Graph neural networks with DAG learning, see Yu et al. ICML 2019
- Continuous optimization framework for single-index model DAG learning, see Yu et al. ICML 2021 (DAGs with no curl)

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- Once we consider interventional data, many new directions/problems suddenly appear!
  1) Hard or soft intervention?
  2) Are the intervened vertices known or unknown to us?
  3) Is the intervention error-prone? (e.g. CRISPR-based gene knockdown techniques are known to have off-target effects)
  4) Are latent variables allowed?

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- Hauser and Bühlmann JMLR 2012 proved: Without hidden variables, given an intervention target sets $\mathcal{I}$, with correct and hard intervention, two DAGs $\mathcal{G}_1$ and $\mathcal{G}_2$ belong to the same $\mathcal{I}$-MEC if and only if for every $I \in \mathcal{I}$, $\mathcal{G}_1(I) \sim \mathcal{G}_2(I)$ in observational sense

R Exercise

- Implementing pcalg

Q2). ID algorithm

- For DAG without hidden variables, identification is easy by Robins' g-formula

$$p(Y(a)=y)=\int \prod_{i\in \mathrm{an}_{\mathcal{G}_{V\setminus A}}(Y)} p_i(v_i|\mathrm{pa}_{\mathcal{G}}(v_i)) \prod_{i\in \mathrm{an}_{\mathcal{G}_{V\setminus A}}(Y)\setminus Y} \mathrm{d}v_i$$

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- The intuition is clear: since CPDAG/MPDAG involves undirected edges, decompose the vertices $V = \cup_{j=1}^{k} \boldsymbol{B}_j$ into smaller units, which is called "bucket" by Perković: in graph-theoretic terms, bucket is the maximally connected component by undirected edges

  For DAGs, buckets are all the singletons of vertices

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- For CPDAG/MPDAG, Perković 2020 UAI gave the identification formula

Theorem 2 (Theorem 3.6 of Perković 2020 UAI)

If there is no path $\langle A, V_1, \cdots, V_k, \cdots, Y\rangle$ from A to Y without edge $V_j \to V_i$ for any $j > i$ in $\mathcal{G}$ starting with $A - \cdots$, then

$$p(Y(a)=y)=\int \prod_{j=1}^{k} p(\boldsymbol{b}_j|\mathrm{pa}_{\mathcal{G}}(\boldsymbol{b}_j))\mathrm{d}\bar{\boldsymbol{b}}$$

where buckets $\boldsymbol{B}_j$'s are buckets of $\mathrm{an}_{\mathcal{G}_{V\setminus A}}(Y)$ and $\bar{\boldsymbol{B}}=\mathrm{an}_{\mathcal{G}_{V\setminus A}}(Y)\setminus Y$

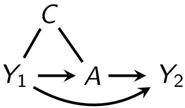
Q2). ID algorithm

Paths like $\langle A, V_1, \cdots, V_k, \cdots, Y \rangle$ without edge $V_j \rightarrow V_i$ for any $j > i$ in $\mathcal{G}$ are called "possibly causal paths". Why excluding "possibly causal paths" that start with $A - \cdots$?

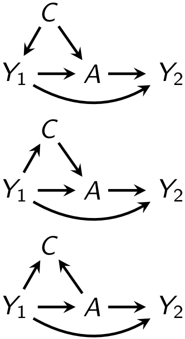
$$A — Y$$

This is a possibly causal path from A to Y, with the first edge being undirected. Why we cannot identify the causal effect of A on Y?

Q2). ID algorithm: Example



CPDAG/MPDAG



MEC of DAGs

$p((Y_1, Y_2)(a))$ identifiable?

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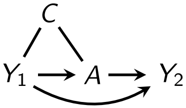

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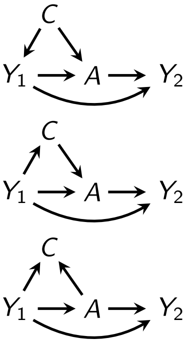
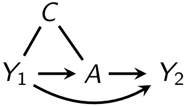


MEC of DAGs

$p((Y_1, Y_2)(a))$ identifiable?

Yes! $A - C - Y_1$, $A \leftarrow Y_1 \rightarrow Y_2$, $A - C - Y_1 \rightarrow Y_2$ not "possibly causal paths" due to $Y_1 \rightarrow A$; $A \rightarrow Y_2$ "possibly causal path" yet not starting with $A - \cdots$

Q2). ID algorithm: Example



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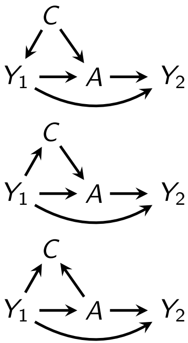


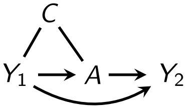
MEC of DAGs

$p((Y_1, Y_2)(a))$ identifiable?

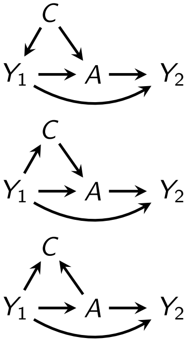
Identification formula: Buckets in $\text{an}_{\mathcal{G}_{V \setminus A}}(Y_1, Y_2) \equiv \{Y_1, Y_2\}$ are $\{Y_1\}$ and $\{Y_2\}$ so

$$p((Y_1, Y_2)(a) = (y_1, y_2)) = p(y_2 | a, y_1) p(y_1)$$

Q2). ID algorithm: Example



CPDAG/MPDAG



MEC of DAGs

$p((Y_1, Y_2)(a))$ identifiable?

Try g-formula with all the DAGs/SWIGs in the MEC

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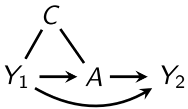

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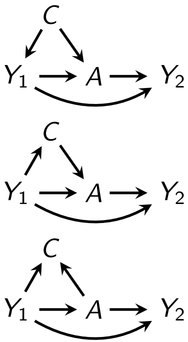
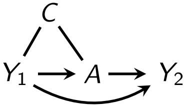


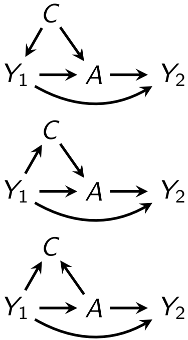
MEC of DAGs

$p((C, Y_1, Y_2)(a))$ identifiable?

Q2). ID algorithm: Example



CPDAG/MPDAG

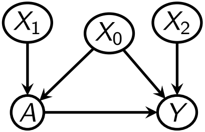


MEC of DAGs

$p((C, Y_1, Y_2)(a))$ identifiable?
No! $A - C$ “possibly causal path” that starts with $A - \cdots$

Q3) Efficient adjustment set (research frontier)

- After ID algorithm outputs yes, a causal query is identifiable but could be *over-identified*, e.g.



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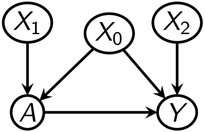


If we want to query $\mathbb{E}[Y(1)]$, there are at least four g-formula:

$$\begin{aligned}
\theta_0 &= \mathbb{E}_{X_0}[\mathbb{E}_Y[Y|X_0, A=1]] \\
\theta_{01} &= \mathbb{E}_{X_0,X_1}[\mathbb{E}_Y[Y|X_0, X_1, A=1]] \\
\theta_{02} &= \mathbb{E}_{X_0,X_2}[\mathbb{E}_Y[Y|X_0, X_2, A=1]] \\
\theta_{012} &= \mathbb{E}_{X_0,X_2}[\mathbb{E}_Y[Y|X_0, X_1, X_2, A=1]]
\end{aligned}$$

So $\{X_0\}$, $\{X_0, X_1\}$, $\{X_0, X_2\}$ and $\{X_0, X_1, X_2\}$ are all valid adjustment sets

- Which one would you pick to use?

Q3) Efficient adjustment set

- Henckel et al. 2019 first formalized the above problem for average causal effects $\mathbb{E}[Y(a)]$ under linear causal models

Q3) Efficient adjustment set

- Henckel et al. 2019 first formalized the above problem for average causal effects $\mathbb{E}[Y(a)]$ under linear causal models

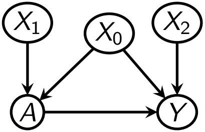
- Rotnitzky and Smucler 2020 demonstrated Henckel et al.'s algorithm also applies to nonlinear models but using very different proof techniques

Q3) Efficient adjustment set

- Henckel et al. 2019 first formalized the above problem for average causal effects $\mathbb{E}[Y(a)]$ under linear causal models

- Rotnitzky and Smucler 2020 demonstrated Henckel et al.'s algorithm also applies to nonlinear models but using very different proof techniques

- Their results have been folklore (with proof) to causal inference researchers for a long time



In the above example, sufficient to only adjust for X_0; further adjusting for X_2 can reduce variance but further adjusting for X_1 can inflate variance

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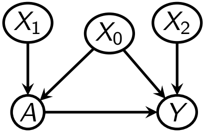

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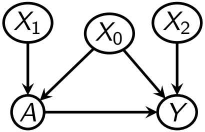

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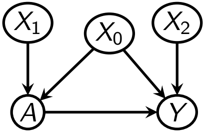

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For future reference, $\{X_0\}$ is called minimal adjustment set because no proper subset of $\{X_0\}$ is still a valid adjustment set

Illustration via linear models

Data generated via linear model

$$\begin{aligned}\mathbb{E}[Y|A,X] &= \beta_{AY}A + \beta_{X_0Y}X_0 + \beta_{X_2Y}X_2 \\ \mathbb{E}[A|X] &= \beta_{X_0A}X_0 + \beta_{X_1A}X_1\end{aligned}$$

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Then the coefficient β_{AY} of A is the causal effect of A on Y

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From basic linear regression $Y \sim A + X_0 + X_1 + X_2$, least square estimator $\widehat{\tau}_{AY}$ satisfies

$$\sqrt{n}\,(\widehat{\tau}_{AY} - \beta_{AY}) \to N\left(0, \frac{\text{var}(Y - \beta_{AY}A - (X_0\ X_1\ X_2)\tau_{XY})}{\text{var}\,(A - (X_0\ X_1\ X_2)\tau_{XA})}\right)$$

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Improving efficiency (reducing variance) from any valid adjustment set

Lemma 1 (Rotnitzky and Smucler, JMLR 2020)

Denote θ_E as the g formula adjusting for the set E. Let B be any valid adjustment set. If there exists a set C s.t. $A \perp\!\!\!\perp C | B$, then $B \cup C$ is also a valid adjustment set, and $\theta_{B \cup C}$ improves over θ_B in the following sense:

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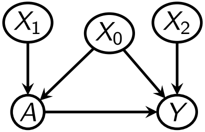

In the above example, $\{X_0\}$ is a valid adjustment set, and $A \perp\!\!\!\perp X_2 | X_0$, so adjusting for $\{X_0, X_2\}$ is better

Comments

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- How to prove? I will use the same example as an illustration: for any valid adjustment set B

$$\begin{aligned}
\sigma_B^2 &= \mathbb{E}\left[\left\{\frac{A}{\Pr(A=1|B)}(Y-\mathbb{E}[Y|A=1,B])+\mathbb{E}[Y|A=1,B]-\theta_B\right\}^2\right] \\
&= \mathbb{E}\left[\frac{A}{\Pr(A=1|B)^2}(Y-\mathbb{E}[Y|A=1,B])^2\right] \\
&\quad + \mathbb{E}_B\left[\{\mathbb{E}[Y|A=1,B]-\theta_B\}^2\right] \\
&= \mathbb{E}\left[\frac{1}{\Pr(A=1|B)}\mathbb{E}\left[(Y-\mathbb{E}[Y|A=1,B])^2|A=1,B\right]\right] \\
&\quad + \mathbb{E}_B\left[\{\mathbb{E}[Y|A=1,B]-\theta_B\}^2\right]
\end{aligned}$$

Comments (continue)

$$
\begin{aligned}
&\sigma_B^2 \\
&= \mathbb{E}\left[\left\{\frac{1}{\Pr(A=1|B)}-1\right\}\mathbb{E}\left[(Y-\mathbb{E}[Y|A=1,B])^2|A=1,B\right]\right] \\
&\quad + \mathbb{E}_B\left[\mathbb{E}\left[(Y-\mathbb{E}[Y|A=1,B])^2|A=1,B\right]\right] + \mathbb{E}_B\left[\left\{\mathbb{E}[Y|A=1,B]-\theta_B\right\}^2\right] \\
&= \mathbb{E}\left[\left\{\frac{1}{\Pr(A=1|B)}-1\right\}\mathrm{var}\left[Y|A=1,B\right]\right] + \mathbb{E}_B[\mathbb{E}[(Y-\theta_B)^2|A=1,B]]
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$$

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\end{aligned}
$$

Similarly

$$
\begin{aligned}
\sigma_{B\cup C}^2 &= \mathbb{E}\left[\left\{\frac{1}{\Pr(A=1|B,C)} - 1\right\}\mathrm{var}\left[Y|A=1,B,C\right]\right] \\
&\quad + \mathbb{E}_{B,C}[\mathbb{E}[(Y-\theta_{B\cup C})^2|A=1,B,C]] \\
\text{(why?)} &= \mathbb{E}\left[\left\{\frac{1}{\Pr(A=1|B)} - 1\right\}\mathbb{E}\left[\mathrm{var}\left[Y|A=1,B,C\right]|A=1,B\right]\right] \\
&\quad + \mathbb{E}_B[\mathbb{E}[(Y-\theta_B)^2|A=1,B]]
\end{aligned}
$$

Comments (continue)

Take the difference:

$$
\begin{aligned}
&\sigma_B^2 - \sigma_{B\cup C}^2 \\
&= \mathbb{E}\Bigg[\left\{\frac{1}{\Pr(A=1|B)} - 1\right\}\left\{\mathsf{var}\left[Y|A=1,B\right] - \mathbb{E}\left[\mathsf{var}\left[Y|A=1,B,C\right]|A=1,B\right] \right. \\
&= \mathbb{E}\left[\left\{\frac{1}{\Pr(A=1|B)} - 1\right\}\mathsf{var}\left\{\mathbb{E}\left[Y|A=1,B,C\right]|A=1,B\right\}\right] \geq 0
\end{aligned}
$$

Q.E.D.

Improving efficiency (reducing variance) from any valid adjustment set

Lemma 2 (Rotnitzky and Smucler, JMLR 2020)

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$$\sigma^2_B \leq \sigma^2_{B \cup C}$$

So combining Lemma 1 and Lemma 2, one can get the benefits of both

Finale of efficient adjustment set

Theorem 3 (Henckel et al. 2019, Rotnitzky and Smucler JMLR 2020)

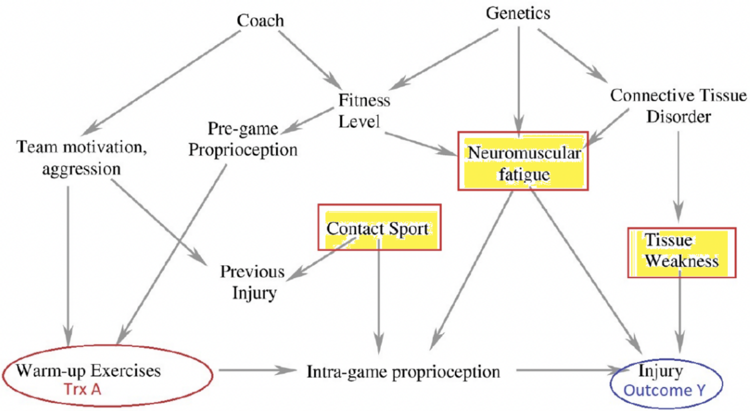
Define

$$O = \left\{ \begin{array}{c} \text{non-descendants of } A \text{ that are also parents of } Y \\ \text{or parents of vertices on the causal path between } A \text{ and } Y \end{array} \right\}$$

Then O is a globally optimal valid adjustment set.

This theorem formalized statisticians' long-standing intuition in the most general way under causal sufficiency

Example



An example stolen from Andrea Rotnitzky's ocis talk

Open problem

- Optimal g-formula for general case in DAG (longitudinal and dynamic regime have been shown to be impossible by Rotnitzky and Smucler, change definition?)

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Causal graph allowing latent variables

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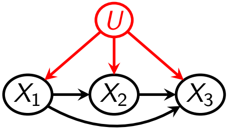
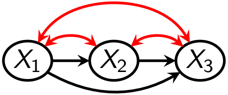
- We will see several examples of ADMGs and mDAGs, but our focus will be on ADMGs

Examples of ADMGs



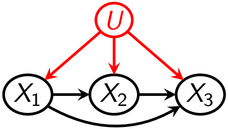
DAG with latent variable



Corresponding ADMG

Introducing bidirected edges, but losing information that all three observables share the same latent variable(s) U

Examples of mDAGs



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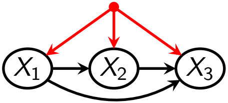

Corresponding mDAG

Introducing hyperedges (the red trident structure in the right graph), increasing the representation complexity (may eventually need a hyperedge with many many endpoints), but keeping more information of the original DAG

Latent projection: Map DAG with latent to ADMG

- Before talking about ADMG, we first define a graphical operation called latent projection

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    - Keep vertices in O and edges between every pair of vertices in O
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  - If there exists a path between X and Y such that the non-endpoints are non-colliders in L, and such that the edge adjacent to the end points are both pointing to the end points, then add $X \leftrightarrow Y$

ADMG

- Definition: A graph $\mathcal{G}^\dagger$ is an ADMG if $\mathcal{G}^\dagger = \mathcal{G}[V]$ for some DAG $\mathcal{G}$

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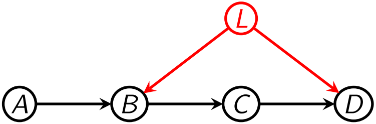
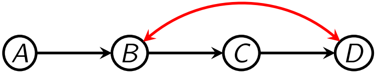
- Example: Verma

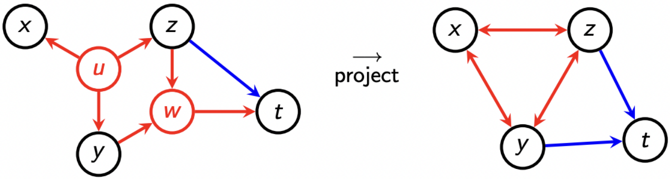


DAG $\mathcal{G}$



ADMG $\mathcal{G}[\{A, B, C, D\}]$

Example



Latent projection leads to an **acyclic directed mixed graph** (ADMG)

Can read off independences with d/m-separation.

The projection preserves the causal structure; Verma and Pearl (1992).

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- Spirtes et al. ["Computation, causation, and discovery" Chapter 6] developed FCI (fast causal inference) algorithm which is sound and complete (after modified by Jiji Zhang) for recovering MAGs

Computer-assisted conjecture of Shpitser for four-vertex ADMG

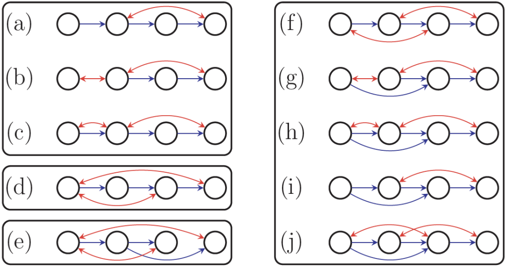

Figure 10: The conjectured equivalence classes among graph patterns (with vertex labeling suppressed) of 4 node ADMGs corresponding to nested Markov models that are strict submodels of the ordinary Markov models given by the same ADMG.

MAG definition

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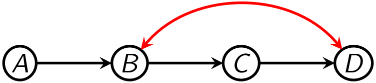

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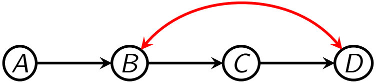
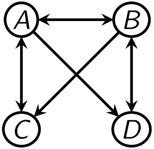
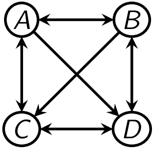


ADMG $\mathcal{G}[\{A, B, C, D\}]$ but not a MAG

- Maximal AGs do not allow for *inducing paths*[1] between any two non-adjacent vertices



Ancestral but not maximal because $\{C - A - B - D\}$ is an inducing path



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Intuition on almost directed cycles and inducing paths

- Why not almost directed cycles?

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To preserve ancestral relationships among the observables

Intuition on almost directed cycles and inducing paths

- Why not almost directed cycles?

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- Why not inducing paths between non-adjacent vertices?

Intuition on almost directed cycles and inducing paths

- Why not almost directed cycles?

To preserve ancestral relationships among the observables

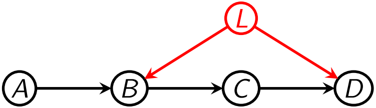
- Why not inducing paths between non-adjacent vertices?

On a DAG, every two non-adjacent vertices have to be d-separated by some other vertices; MAG needs to preserve such a property because no observed data can distinguish whether or not the edge between C and D exists

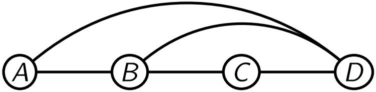
From DAG to MAG

Two step procedure: input a DAG $\mathcal{G}$ output its MAG $\mathcal{M}$

(1) Two observables A and B in $\mathcal{G}$ are adjacent in $\mathcal{M}$ if and only if there is an inducing path between A and B relative to the hidden vertices (i.e. hidden vertices on the path can be non-colliders)



DAG with hidden L



MAG from $\mathcal{G}$

- every direct edge between two observables is an inducing path relative to L
- $A \rightarrow B \leftarrow L \rightarrow D$ inducing path rel. to L: B is a collider and $B \in \text{an}(D)$
- $B \rightarrow L \leftarrow D$ inducing path rel. to L

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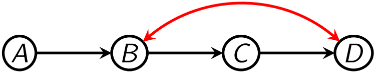
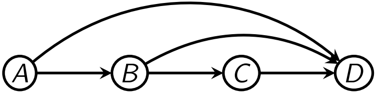
(2) Orient $A \rightarrow B$ if $A = \text{an}(B)$, $A \leftarrow B$ if $B = \text{an}(A)$, and $A \leftrightarrow B$ if otherwise



ADMG $\mathcal{G}[\{A, B, C, D\}]$ but not a MAG



MAG from $\mathcal{G}$

MAGs and m-separation

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MAGs and m-separation

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- they are called m-separation

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- MEC of MAGs can be represented by PAGs (partial ancestral graphs), analogous to CPDAGs

Supplement: Discriminating path

In a MAG, a path p between A and B, e.g. $p = (A, \cdots, W, V, B)$ is a discriminating path for V if

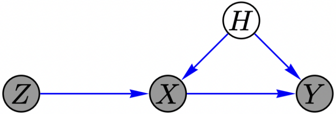
1. p includes at least three edges (and of course, at least four vertices)
2. V is adjacent to one endpoint on p and in the above case, B
3. A and B non-adjacent, and every vertex between A and V is a collider on p and is a parent of B

The reason we have to consider discriminating path for MAG is that when p is a discriminating path, it behaves as a v-structure between A and B in terms of the triple (W, V, B) in the following sense:

1. (W, V, B) is a non-collider if and only if every set m-separating A and B contains V
2. (W, V, B) is a collider with $W \to B$ or $W \leftarrow B$ if and only if every set m-separating A and B does not contain V

Difference (I)

There can be observed variables which are not adjacent, but for which no subset of the other observed variables d-separates them.



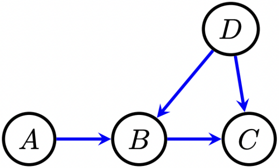
Here H is unobserved.

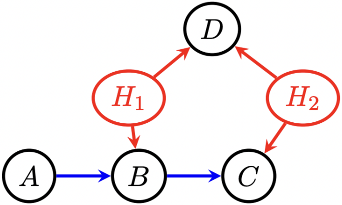
Z and Y are d-connected given $\emptyset$.

Z and Y are d-connected given $\{X\}$.

Difference (II)

For DAGs without hidden variables, *same adjacencies* and *same unshielded colliders* were necessary and sufficient for equivalence ⇒ only need to look at structures involving at most 3 vertices.





These graphs are not Markov equivalent over the observed margin.

A is d-separated from C given $\{B, D\}$ in the left graph.
A is d-separated from C given $\{B\}$ in the right graph.

⇒ Need to look at more complex structures.

Q1) Causal discovery with latent variables: FCI algorithm

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- Spirtes et al. also proved as sample size $n \to \infty$, the output of FCI converges to the MEC of the true underlying MAG, under faithfulness

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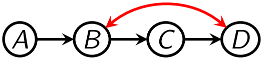
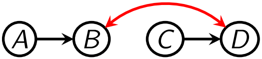


$\overset{\text{reweight by } \frac{\widetilde{f}(c)}{f(c|b)}}{\Rightarrow}$



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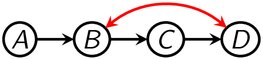

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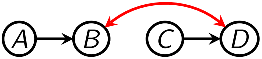

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- MAGs fail to preserve such dormant independences, or nested Markov properties

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- It culminated at the paper by Caroline Uhler et al. in 2014, who ingeniously applies classical results from algebraic geometry to this problem

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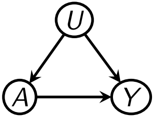

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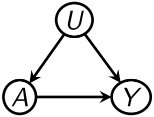

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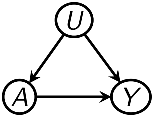

Faithfulness: $A \perp\!\!\!\perp Y \Rightarrow A$ and Y are d-separated

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- Intuitively speaking, in the above DAG, faithfulness rules out the possibility that $A \leftarrow U \rightarrow Y$ effect somehow cancel out the effect $A \rightarrow Y$ when looking at the marginal dependence between A and Y!

Is it reasonable to assume faithfulness?

The answer is quite mixed.

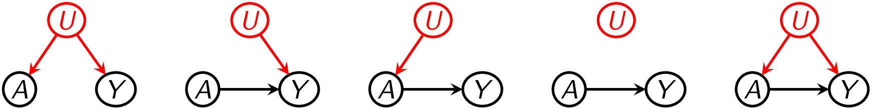
On one hand, for probability distributions parameterized by finite-dimensional parameters, the set of distributions Markov to a DAG but unfaithful to that DAG has Lebesgue measure 0 (Meek. UAI 1995)!

On the other hand... let's see towards the end of this section

Is consistency enough?

We will consider the following example: observe (A, Y) and U could be unmeasured but we assume the background knowledge U precedes A and Y and A precedes Y. So we have the following 8 potential DAGs:

Subset 1: $A \not\perp\!\!\!\perp Y$, FCI reports “don’t know”



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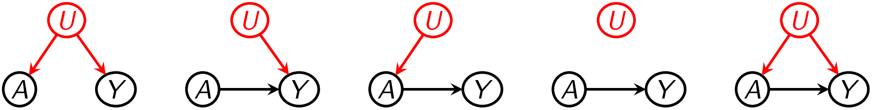

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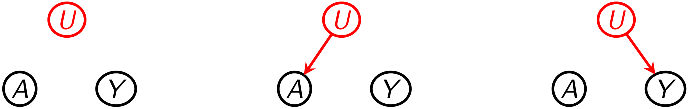

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  For the true but unknown distribution $\mathcal{P}$, given an error tolerance $\epsilon > 0$, we can find a large integer $N(\epsilon, \mathcal{P}) \equiv N > 0$, such that for every $n > N$, the sum of type-I and type-II errors of testing H_0 is below ϵ

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- ϵ-δ translation of uniform consistency:
  For every distribution, given an error tolerance $\epsilon > 0$, we can find a large integer $N(\epsilon) \equiv N > 0$, such that for every $n > N$, the sum of type-I and (non-local) type-II errors of testing H_0 is below ϵ

Why uniform consistency is what we need?

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Therefore, a meaningful statistical theory should be uniform rather than point-wise (one can also connects such disparity to Hodges' estimator in classical statistical theory)

If U is observed and U is categorical

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- If U is categorical, then testing $H_0 : Y \perp\!\!\!\perp A | U$ is equivalent to testing finitely many marginal independences, for which there may exist uniformly consistent tests (e.g. dcorr, BETs, ...)

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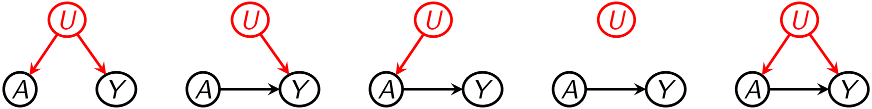
- If the level of U is large compared to sample size (e.g. high-dimensional multinomial), then it again becomes hard

If U is latent but no faithfulness for U is assumed

Subset 1: $A \not\perp\!\!\!\perp Y$, FCI reports "don't know"



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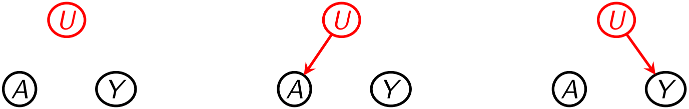

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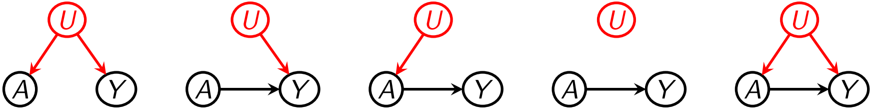

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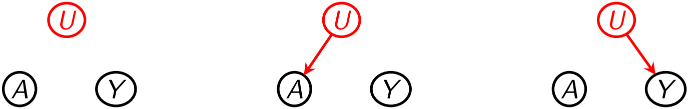

Without faithfulness, even no consistent method because $A \perp\!\!\!\perp Y$ can be compatible with the last DAG in Subset 1, so
H_0 : no causal effect from A to Y will be erroneously accepted

If U is latent and faithfulness is assumed

With faithfulness, we rule out the last DAG in Subset 1, so H_0 : no causal effect from A to Y can be safely rejected or accepted...

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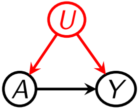

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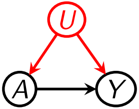

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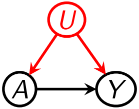

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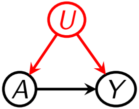

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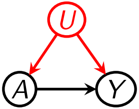

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Because a valid independence test must not reject H_0 with high probability when H_0 is indeed correct and $O(n^{-1/2})$ is the finite sampling error of such a test

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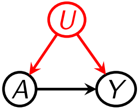
- λ-strong faithfulness: also remove distributions for which the marginal dependence between A and Y is $O\{\lambda\}$



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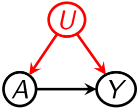

- Under $(n/\log n)^{1/2}$-strong faithfulness, one can get uniform consistency

Caroline Uhler's counter-argument

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- When p is moderately large, this lower bound gets close to 1

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- But the above does not rule out the possibility of high-quality causal discovery given (1) sufficient background knowledge, (2) correct modeling assumption, and (3) possibility of high-quality randomization or perturbation

Final words about causal discovery

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- Combining more and more interventional data might be promising!

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  - Later, Shpitser and Pearl proved ID algorithm is complete (like necessary): if ID outputs "not identifiable", then neither do any other algorithms
    Shpitser, Pearl. Identification of Joint Interventional Distributions in Recursive Semi-Markovian Causal Models. AAAI 2006.
    Shpitser, Pearl. Complete identification methods for the causal hierarchy. JMLR 2008.

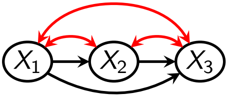
A very brief introduction to theory of ADMGs

CADMGs

ADMGs $\mathcal{G}(V, E)$ with E containing bi-directed edges

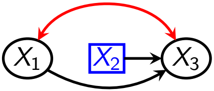
CADMGs (Conditional ADMGs) $\mathcal{G}(V', W, E')$, with V' and W denoting the "random" and "fixed" vertices respectively:

1. $V = V' \cup W$
2. No edges among vertices in W
3. Removing all the arrows into W
4. Turning circles around vertices in W into squares



ADMG:
$\mathcal{G}(\{X_1, X_2, X_3\}, E)$

$\overset{\text{Fixing } X_2}{\Longrightarrow}$



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ID algorithm for ADMG

- Question: Is $p(Y(a) = y)$ or $\mathbb{E}[Y(a)]$ identifiable given an ADMG $\mathcal{G}$ and (subsets of) vertices A and Y?

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- Some papers to read:
  Richardson, Evans, Robins, Shpitser. 2017
  Bhattacharya, Nabi, Shpitser. 2020

```
function $\mathbf{ID}(\mathbf{y}, \mathbf{x}, \mathrm{P}, \mathrm{G})$
INPUT: $\mathbf{x}$,$\mathbf{y}$ value assignments, P a probability distribution, G a causal diagram.
OUTPUT: Expression for $P_{\mathbf{x}}(\mathbf{y})$ in terms of P or FAIL(F,F').

  1 if $\mathbf{x} = \emptyset$ return $\sum_{\mathbf{v} \setminus \mathbf{y}} P(\mathbf{v})$.

  2 if $\mathbf{V} \setminus An(\mathbf{Y})_G \neq \emptyset$
    return $\mathbf{ID}(\mathbf{y}, \mathbf{x} \cap An(\mathbf{Y})_G, \sum_{\mathbf{v} \setminus An(\mathbf{Y})_G} P, G_{An(\mathbf{Y})})$.

  3 let $\mathbf{W} = (\mathbf{V} \setminus \mathbf{X}) \setminus An(\mathbf{Y})_{G_{\overline{\mathbf{X}}}}$.
    if $\mathbf{W} \neq \emptyset$, return $\mathbf{ID}(\mathbf{y}, \mathbf{x} \cup \mathbf{w}, P, G)$.

  4 if $C(G \setminus \mathbf{X}) = \{S_1, ..., S_k\}$
    return $\sum_{\mathbf{v} \setminus (\mathbf{y} \cup \mathbf{x})} \prod_i \mathbf{ID}(s_i, \mathbf{v} \setminus s_i, P, G)$.

    if $C(G \setminus \mathbf{X}) = \{S\}$

      5 if $C(G) = \{G\}$, throw $\mathbf{FAIL}(G, G \cap S)$.
      6 if $S \in C(G)$ return $\sum_{s \setminus \mathbf{y}} \prod_{\{i|V_i \in S\}} P(v_i | v_\pi^{(i-1)})$.
      7 if $(\exists S')S \subset S' \in C(G)$ return $\mathbf{ID}(\mathbf{y}, \mathbf{x} \cap S',$
        $\prod_{\{i|V_i \in S'\}} P(V_i | V_\pi^{(i-1)} \cap S', v_\pi^{(i-1)} \setminus S'), G_{S'})$.
```

Figure 4: A complete identification algorithm. **FAIL** propagates through recursive calls like an exception, and returns the hedge which witnesses non-identifiability. $V_\pi^{(i-1)}$ is the set of nodes preceding V_i in some topological ordering π in G.

Preparatory definitions

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Proof.

For DAG, $\text{dis}_{\mathcal{G}}(v) = \{v\}$. Then

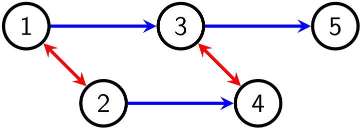
$$\text{mb}_{\mathcal{G}}(v) \equiv \text{pa}_{\mathcal{G}}(v) \cup (\{v\} \setminus \{v\}) \equiv \text{pa}_{\mathcal{G}}(v).$$

□

So Markov Blanket generalizes Parent Set in DAG to ADMG

Districts

Define a **district** in a C/ADMG to be maximal sets connected by bi-directed edges:



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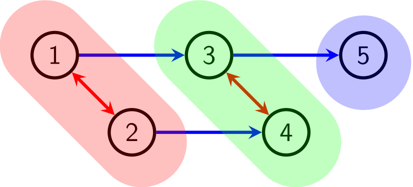

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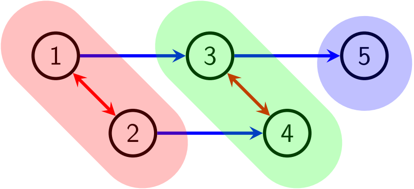

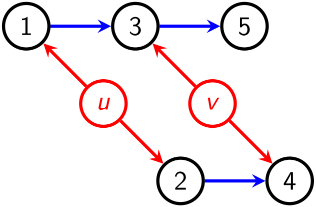

$$\sum_{u,v} p(u)\,p(x_1 \mid u)\,p(x_2 \mid u) \quad p(v)\,p(x_3 \mid x_1, v)\,p(x_4 \mid x_2, v) \quad p(x_5 \mid x_3)$$

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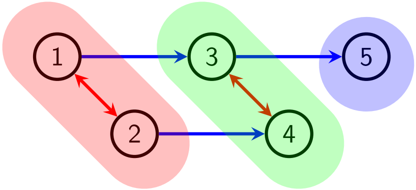

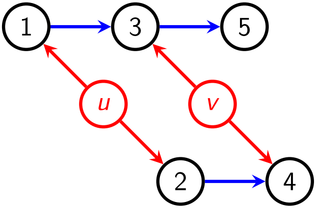

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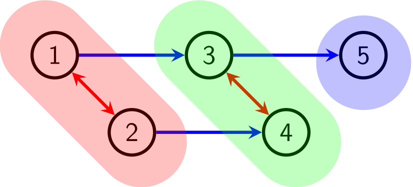

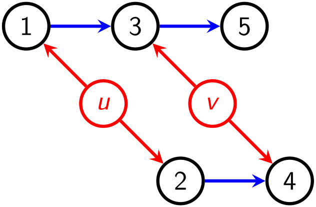

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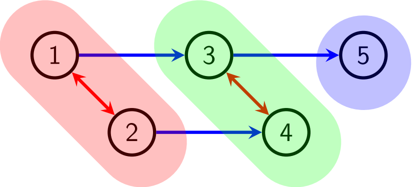

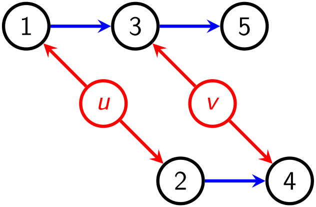

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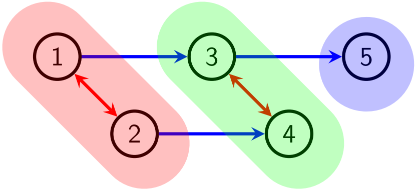

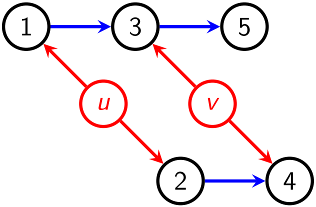

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$$= \prod_i q_{D_i}(x_{D_i} \mid x_{\mathrm{pa}(D_i)\setminus D_i})$$

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- Fixable vertices

$$F(\mathcal{G}) := \{v \in V : \mathrm{dis}_{\mathcal{G}}(v) \cap \mathrm{de}_{\mathcal{G}}(v) = \{v\}\}$$

In words, v is fixable if no vertex $x \neq v$ s.t.

$$v \leftrightarrow \cdots \leftrightarrow x \text{ and } v \rightarrow \cdots \rightarrow x$$

Trivial implication: singletons are always fixable and vertices in a DAG are always fixable

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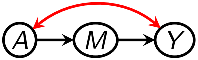
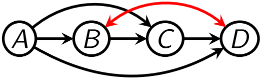
- Examples:



Front door: $\mathcal{D}_{\mathcal{G}} = \{\{M\},\{A,Y\}\}$,
$F(\mathcal{G}) = \{M,Y\}$



Verma: $\mathcal{D}_{\mathcal{G}} = \{\{A\},\{B,D\},\{C\}\}$,
$F(\mathcal{G}) = \{A,C,D\}$

Fixing operation: Graphical operation

For every $r \in F(\mathcal{G})$, graphically fixing operation is defined as

$$\phi_{\{r\}}(\mathcal{G}) := \mathcal{G}(V \setminus \{r\}, W \cup \{r\}, E')$$

where E' is edge set in the original ADMG $\mathcal{G}$ by removing all edges pointing towards $\{r\}$

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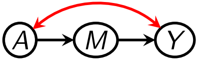
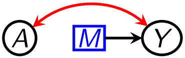


$F(\mathcal{G}) = \{M, Y\}$

$\xrightarrow{\text{Fixing } M}$



$\phi_M(\mathcal{G}), F(\phi_M(\mathcal{G})) = \{A, Y\}$

Fixing operation: Algebraic operation

$p(x_V|x_W)$ is the distribution of all the random vertices of a CADMG $\mathcal{G}(V, W, E)$; fixing a vertex $\{r\}$ means

$$\phi_{\{r\}}(p(x_V|x_W); \mathcal{G}) = \frac{p(x_V|x_W)}{p(x_r|x_{\mathrm{mb}_{\mathcal{G}}(r)})}$$

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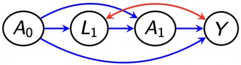
If $r \in F(\mathcal{G})$, then $\phi_{\{r\}}(p(x_V|x_W); \mathcal{G}) \equiv p(x_{V\setminus\{r\}}|x_{W\cup\{r\}})$

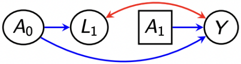
Sequential randomized trial example

Example: Sequential Randomization

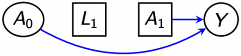
$\mathcal{G}$



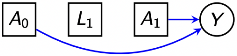
$\phi_{A_1}(\mathcal{G})$



$\phi_{L_1} \circ \phi_{A_1}(\mathcal{G})$

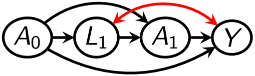


$\phi_{A_0} \circ \phi_{L_1} \circ \phi_{A_1}(\mathcal{G})$



This establishes that $P(Y \mid \text{do}(A_0, A_1))$ is identified.

Using fixing to derive the ID formula



$p(a_0, \ell_1, a_1, y) \equiv$
$p(a_0)p(a_1|a_0, \ell_1)q(\ell_1, y|a_0, a_1)$,
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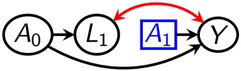
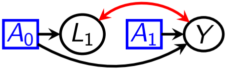
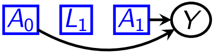


Fix A_1: $\dfrac{p(a_0, \ell_1, a_1, y)}{p(a_1|a_0, \ell_1)} \equiv$
$p(a_0)q(\ell_1, y|a_0, a_1) =: p^{(1)}(a_0, \ell_1, y|a_1)$

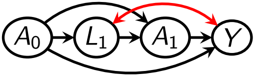


Fix A_0: $\dfrac{p^{(1)}(a_0, \ell_1, y|a_1)}{p(a_0)} \equiv q(\ell_1, y|a_0, a_1) =:$
$p^{(2)}(\ell_1, y|a_0, a_1)$



Fix L_1:
$\frac{p^{(2)}(\ell_1, y|a_0, a_1)}{q(\ell_1|a_0, a_1, y)} = q(y|a_0, a_1) =: p^{(3)}(y|a_0, a_1)$

Using fixing to derive the ID formula



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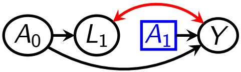

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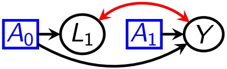

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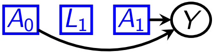

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$p^{(3)}(y|a_0, a_1) = q(y|a_0, a_1) = \int_{\ell_1} q(\ell_1, y|a_0, a_1)\mathrm{d}\ell_1$ is the g-formula because

$$q(\ell_1, y|a_0, a_1) = \frac{p(a_0, \ell_1, a_1, y)}{p(a_0)p(a_1|a_0, \ell_1)} = p(\ell_1|a_0)p(y|a_0, \ell_1, a_1)$$

The orders of fixing operations don't matter

- Comparing the above two slides, you will discover that they used two different fixing sequences but both lead to the same ID formula

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- This is the main result that Richardson, Evans, Robins and Shpitser proved in 2017 (Theorem 32 of RERS17): otherwise using fixing operation would have been an absurd idea!

Reachable Subgraphs and Intrinsic Sets

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- The set of all intrinsic sets in $\mathcal{G}$ is denoted as $\mathcal{I}(\mathcal{G})$

Tian's ID algorithm

Generalizing the above special case, Tian's ID algorithm can be formulated as follows

Theorem 4 (Theorem 49 of RERS17)

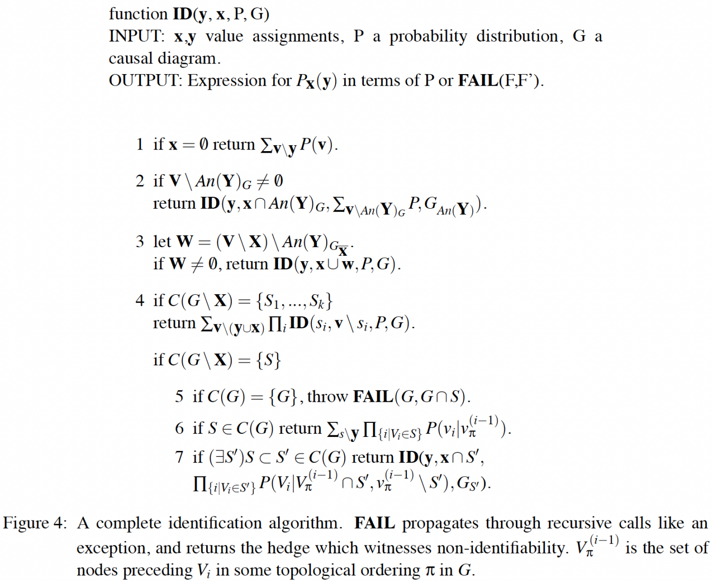
Given an ADMG $\mathcal{G}(V, E) \equiv \mathcal{G}$ and two disjoint subsets $A, Y \subseteq V$, let $\overleftarrow{Y} := \mathrm{an}_{\mathcal{G}_{V\setminus A}}(Y)$. If $\mathcal{D}(\mathcal{G}_{\overleftarrow{Y}}) \subseteq \mathcal{I}(\mathcal{G})$, then

$$\begin{aligned} p(X_Y(x_A) = x_Y) &= \int_{x_{\overleftarrow{Y}\setminus Y}} \prod_{D \in \mathcal{D}(\mathcal{G}_{\overleftarrow{Y}})} p(X_D(x_{\mathrm{pa}_{\mathcal{G}}(D)\setminus D}) = x_D) \mathrm{d}x_{\overleftarrow{Y}\setminus Y} \\ &= \int_{x_{\overleftarrow{Y}\setminus Y}} \prod_{D \in \mathcal{D}(\mathcal{G}_{\overleftarrow{Y}})} \phi_{V\setminus D}(p(x_V); \mathcal{G}) \mathrm{d}x_{\overleftarrow{Y}\setminus Y}. \end{aligned} \tag{1}$$

If not, there exists $D \in \mathcal{D}(\mathcal{G}_{\overleftarrow{Y}})$ not in the intrinsic sets and $p(X_Y(x_A) = x_Y)$ is unidentifiable.

Think about the following question: Try to translate the above theorem using SWIGs

For the sake of comparison



function **ID**($\mathbf{y}$, $\mathbf{x}$, P, G)
INPUT: $\mathbf{x}$,$\mathbf{y}$ value assignments, P a probability distribution, G a causal diagram.
OUTPUT: Expression for $P_{\mathbf{x}}(\mathbf{y})$ in terms of P or **FAIL**(F,F').

```
1 if $\mathbf{x} = \emptyset$ return $\sum_{\mathbf{v} \setminus \mathbf{y}} P(\mathbf{v})$.
2 if $\mathbf{V} \setminus An(\mathbf{Y})_G \neq \emptyset$
  return $\mathbf{ID}(\mathbf{y}, \mathbf{x} \cap An(\mathbf{Y})_G, \sum_{\mathbf{V} \setminus An(\mathbf{Y})_G} P, G_{An(\mathbf{Y})})$.
3 let $\mathbf{W} = (\mathbf{V} \setminus \mathbf{X}) \setminus An(\mathbf{Y})_{G_{\overline{\mathbf{X}}}}$.
  if $\mathbf{W} \neq \emptyset$, return $\mathbf{ID}(\mathbf{y}, \mathbf{x} \cup \mathbf{w}, P, G)$.
4 if $C(G \setminus \mathbf{X}) = \{S_1, ..., S_k\}$
  return $\sum_{\mathbf{v} \setminus (\mathbf{y} \cup \mathbf{x})} \prod_i \mathbf{ID}(s_i, \mathbf{v} \setminus s_i, P, G)$.
  if $C(G \setminus \mathbf{X}) = \{S\}$
    5 if $C(G) = \{G\}$, throw $\mathbf{FAIL}(G, G \cap S)$.
    6 if $S \in C(G)$ return $\sum_{s \setminus \mathbf{y}} \prod_{\{i|V_i \in S\}} P(v_i | v_\pi^{(i-1)})$.
    7 if $(\exists S') S \subset S' \in C(G)$ return $\mathbf{ID}(\mathbf{y}, \mathbf{x} \cap S',$
      $\prod_{\{i|V_i \in S'\}} P(V_i | V_\pi^{(i-1)} \cap S', v_\pi^{(i-1)} \setminus S'), G_{S'})$.
```

Figure 4: A complete identification algorithm. **FAIL** propagates through recursive calls like an exception, and returns the hedge which witnesses non-identifiability. $V_\pi^{(i-1)}$ is the set of nodes preceding V_i in some topological ordering π in G.

Intuition of Tian's ID algorithm

- Only the subgraph $\mathcal{G}^*$ of the ancestors of Y, with the causal path to Y not including A, needs to be considered for identifying $p(X_Y(x_A) = x_Y)$

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- Then

$$p(X_Y(x_A) = x_Y) = \int_{x_{\overleftarrow{Y} \setminus Y}} \prod_{D \in \mathcal{D}(\mathcal{G}_{\overleftarrow{Y}})} p(X_D(x_{\mathrm{pa}_{\mathcal{G}}(D) \setminus D}) = x_D) \mathrm{d}x_{\overleftarrow{Y} \setminus Y}$$

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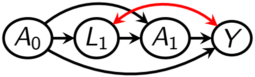
- Find if $p(X_D(x_{\mathrm{pa}_{\mathcal{G}}(D) \setminus D}) = x_D)$ is identified

Exercise 1: Verma or sequential randomized trial

$\mathcal{G}$



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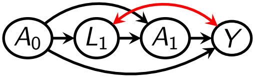
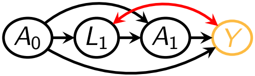


$\overleftarrow{Y} = \mathrm{an}_{\mathcal{G}_{V \setminus A}}(Y)$



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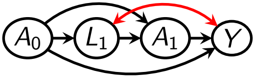

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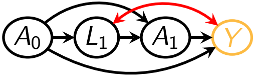

So $\mathcal{D}(\mathcal{G}_{\overleftarrow{Y}}) = \{Y\}$, $\text{pa}_{\mathcal{G}}(Y) = \{A_0, A_1\}$ and

$$\begin{aligned} p(X_Y(x_{A_0}, x_{A_1}) = x_Y) &= \prod_{D \in \mathcal{D}(\mathcal{G}_{\overleftarrow{Y}})} p(X_D(x_{\text{pa}_{\mathcal{G}}(D)\setminus D}) = x_D) \\ &= p(X_Y(x_{A_0}, x_{A_1}) = x_Y) \end{aligned}$$

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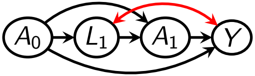

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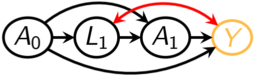

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Identified?

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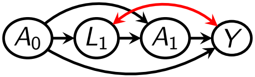

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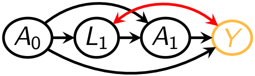

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\end{aligned}
$$

Identified? Is $\{Y\}$ an intrinsic set?

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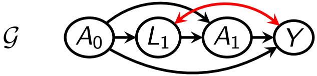

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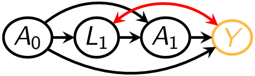

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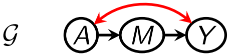
$$= p(X_Y(x_{A_0}, x_{A_1}) = x_Y)$$

Identified? Is $\{Y\}$ an intrinsic set? Yes! We have seen $\{Y\}$ is reachable by fixing A_1, A_0, L_1

Exercise 2: Front door



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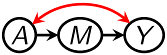
$\mathcal{G}$



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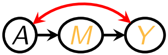

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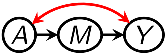

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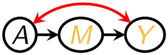

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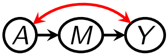

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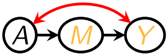

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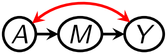

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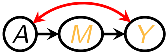

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$$= \int_{x_M} p(X_Y(x_M) = x_Y) p(X_M(x_A) = x_M) \mathrm{d}x_M$$

Identified? Are $\{M\}$ and $\{Y\}$ intrinsic sets?

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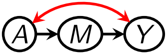

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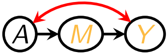

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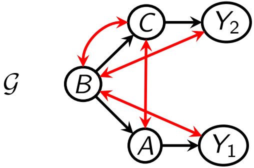
$$
\begin{aligned}
p(X_Y(x_A) = x_Y) &= \int_{x_{\overleftarrow{Y}\setminus Y}} \prod_{D \in \mathcal{D}(\mathcal{G}_{\overleftarrow{Y}})} p(X_D(x_{\mathrm{pa}_{\mathcal{G}}(D)\setminus D}) = x_D)\mathrm{d}x_{\overleftarrow{Y}\setminus Y} \\
&= \int_{x_M} p(X_Y(x_M) = x_Y) p(X_M(x_A) = x_M)\mathrm{d}x_M
\end{aligned}
$$

Identified? Are $\{M\}$ and $\{Y\}$ intrinsic sets? Yes for M; Yes for Y by fixing M and A

Exercise 3: causal effect of A on $Y = \{Y_1, Y_2\}$



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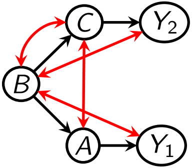
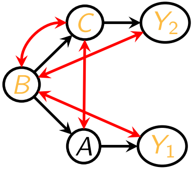
$\mathcal{G}$



$\overleftarrow{Y} = \mathrm{an}_{\mathcal{G}_{V \setminus A}}(Y)$



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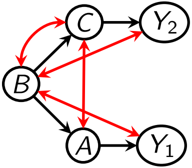

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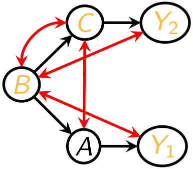

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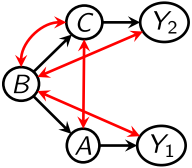

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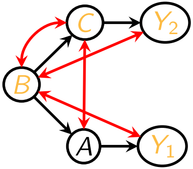

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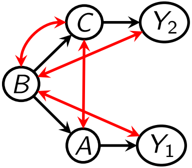

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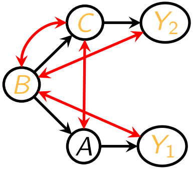

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Identified? Is D intrinsic? No! Because A is not fixable in $\mathcal{G}$!

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- Runge proposed a sound and complete algorithm for ADMGs

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  - Differentiable causal discovery/structure learning with linear SEM allowing latent variables to recover certain ADMGs
  - Given an ADMG, whether a causal query is identifiable
  - If over-identified, which formula should we use (Shpitser's group is working on a symbolic computation software just like mathematica or maple)

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- Except probability and graph/combinatorics, algebraic geometry is another subfield of pure math that is extremely useful for causal inference
- We will discuss inequality constraints in next chapter

Any Questions?