Causal Inference Methods in Data Science
Lecture 2: More on Causal Identifications, DAGs, SWIGs, Sensitivity Analysis, Mediation

Lin Liu

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two nice online seminars

online causal inference seminar (ocis):
https://sites.google.com/view/ocis/

the Gary Chamberlain online seminar in econometrics:
https://www.chamberlainseminar.org/

Definition of a confounder

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C is a confounder relative to the effect of A on Y if there exists a set of pre-treatment (A) covariates X such that

(1) $Y(a) \perp\!\!\!\perp A|X, C$

(2) But there is no proper subset $Z \subsetneq (X, C)$ such that $Y(a) \perp\!\!\!\perp A|Z$

Rigorous derivation of ATE identification

We need the following ID (identification) conditions

1. Consistency: $Y = AY(1) + (1 - A)Y(0)$
2. Positivity: $0 < \Pr(A = a|X) < 1$ almost surely
3. No unmeasured confounding/ignorability: $Y(a) \perp\!\!\!\perp A|X, \ \forall a \in \{0, 1\}$

Static g-formula of $\mathbb{E}[Y(1)]$ in observational studies under ignorability

$$\begin{aligned}\mathbb{E}[Y(1)] &= \mathbb{E}[\mathbb{E}[Y(1)|X]] \\ &\overset{2,3}{=} \mathbb{E}[\mathbb{E}[Y(1)|X, A = 1]] \\ &\overset{1}{=} \mathbb{E}[\mathbb{E}[Y|X, A = 1]]\end{aligned}$$

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IPW of $\mathbb{E}[Y(1)]$ in observational studies under ignorability

$$
\begin{aligned}
\mathbb{E}[Y(1)] &= \mathbb{E}[\mathbb{E}[Y|X, A = 1]] \\
&= \int \mathbb{E}[Y|X = x, A = 1]p(x)\mathrm{d}x \\
&\overset{2}{=} \int \frac{y}{p(A = 1|X = x)} p(y|X = x, A = 1)p(A = 1|X = x)p(x)\mathrm{d}y\mathrm{d}x \\
&= \int \frac{ay}{p(A = 1|X = x)} p(y|X = x, A = 1)p(A = a|X = x)p(x)\mathrm{d}y\mathrm{d}a\mathrm{d}x \\
&= \mathbb{E}\left[\frac{AY}{p(A = 1|X)}\right]
\end{aligned}
$$

A question to think about

Can you derive the IPW formula without using g-formula?

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  - The rest is statistics and computing (optimization)

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- Mediation analysis
- Next chapter is on dynamic causal inference, and mediation analysis is also a sort of dynamic causal inference

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- DAGs are very convenient to represent background causal knowledge
- A causes Y: in word, it means

$$\begin{cases} \text{i)} & A \text{ precedes } Y \text{ in time ordering} \\ \text{ii)} & \text{a change in } Y \text{ is only due to a change in } A \end{cases}$$

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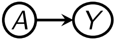

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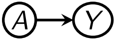

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  A precedes Y in time in human language

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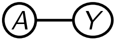

Figure: Either A causes Y or Y causes A

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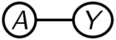

Figure: Either A causes Y or Y causes A

- Graphs with only (un)directed edges: (un)directed graphs

- Graphs with both directed and undirected edges: partially directed graphs

Definition of DAGs: More terminologies (paths family)

- *path* between nodes X and Y on $\mathcal{G}$
  any sequence of **distinct nodes** $(X, V_1, \cdots, V_k, Y)$, $k \geq 0$, such that any two successive nodes are connected by an edge
  e.g. $X \rightarrow V_1 - V_2 \leftarrow \cdots \leftarrow V_k - Y$

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- $X = \text{an}_{\mathcal{G}}(Y) \equiv \text{an}(Y)$ if there is any causal path from X to Y

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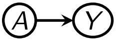
- On a path, if $X_i \rightarrow X_j \leftarrow X_k$, then X_j is a collider

A quick quiz: is it possible $X = \mathrm{an}(Y)$ and $X = \mathrm{de}(Y)$ in a DAG?

Examples of DAGs

- A causes Y



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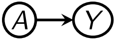
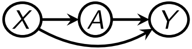


- A causes Y and X is a common cause (confounder) of A and Y



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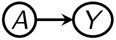

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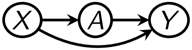
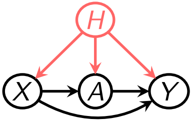


- Observational studies with unmeasured/latent confounders:



One potential disadvantage of DAG: unless we are crystally clear about which important confounders are unmeasured, it is impossible to draw such a DAG

From DAGs without latent variables to probability distribution (data)

Markov factorization of DAGs:

Definition 1

A probability density function f over the variables $\boldsymbol{V}$ is consistent with DAG $\mathcal{G} = (\boldsymbol{V}, \boldsymbol{E})$ ($\boldsymbol{E}$ is the set of all directed edges) if it factorizes according to the following rule:

$$f(\boldsymbol{v}) = \prod_{V_i \in \boldsymbol{V}} f\left(v_i | \mathrm{pa}_{\mathcal{G}}(v_i)\right) \tag{1}$$

IPW, interventional distribution and do calculus

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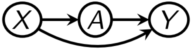

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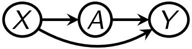

By Markovian factorization of DAG,

$$p(X, A, Y) = p(Y|X, A)p(A|X)p(X)$$

so when data is drawn from the above DAG, A is drawn from a distribution with pdf/pmf $p(A|X)$

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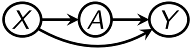

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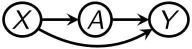

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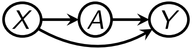

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The interventional distribution:

$$p(X, Y(1)) \equiv p(X, \mathrm{do}(A=1), Y) = p(Y|X, \mathrm{do}(A=1))\mathbb{1}\{A=1\}p(X)$$

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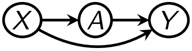

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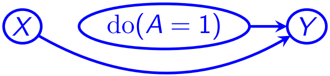

IPW, interventional distribution and do calculus

From the observed data distribution to the interventional distribution?

$$p(X, A, Y) = p(Y|X, A)p(A|X)p(X) \text{ vs.}$$
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Change of probability distribution

$$p(X, \mathrm{do}(A = 1), Y) = \frac{\mathbb{1}\{A = 1\}}{p(A|X)} p(X, A, Y) = \frac{A}{p(A|X)} p(X, A, Y)$$

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So

$$\mathbb{E}[Y(1)] \equiv \mathbb{E}[Y|\mathrm{do}(A = 1)] = \mathbb{E}\left[\frac{A}{p(A|X)} Y\right]$$

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  2. If $\boldsymbol{Z}$ blocks all paths between some disjoint subsets $\boldsymbol{X}$ and $\boldsymbol{Y}$, then $\boldsymbol{X}$ is d-separated from $\boldsymbol{Y}$ given $\boldsymbol{Z}$, denoted as $\boldsymbol{X} \perp_{\mathcal{G}} \boldsymbol{Y} | \boldsymbol{Z}$

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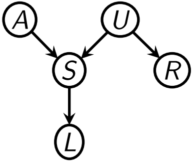

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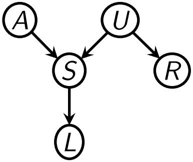
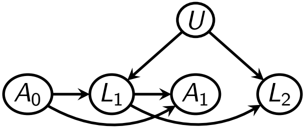


$S \perp_{\mathcal{G}} R | U$, $A \not\perp_{\mathcal{G}} U | S$, $A \not\perp_{\mathcal{G}} U | L$

Another example: due to Verma



quiz: $\boldsymbol{X} = (A_0, A_1)$, $\boldsymbol{Y} = L_2$. Are $\boldsymbol{X}$ and $\boldsymbol{Y}$ d-separated given L_1?

connection to science: can you think of a scientific story for Verma's constraint?

d-separation and statistical independence

- When $\boldsymbol{X} \perp_{\mathcal{G}} \boldsymbol{Y}|\boldsymbol{Z}$, then for every distribution Markov factorized according to $\mathcal{G}$, $\boldsymbol{X} \perp\!\!\!\perp \boldsymbol{Y}|\boldsymbol{Z}$, where $\perp\!\!\!\perp$ stands for "statistical independence"
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- Markov factorization implies:

Conditional on its direct causes/parents, node X is independent of any node it does not cause (any non-descendant of X)

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Does not always hold, how to proceed? Making such an assumption – called "faithfulness"; we will come back to this when we talk about causal discovery/structure learning (a lot of works on linear models done by the statistics group from ETH Zürich)

Causal DAG for RCT

Can you draw a causal DAG for RCT?

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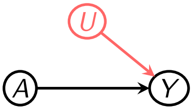

Causal DAG for observational studies under ignorability

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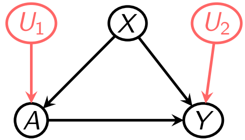

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- But they are not very convenient to use because there are no counterfactuals on DAG!
  - When causal graph theory was first introduced into the statistics community, this is the very reason why Donald Rubin and Guido Imbens (economist) are emphatically against the adoption of DAG

Backdoor criterion: identification rule 1 using DAG

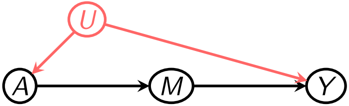
A set of covariates X satisfies the backdoor criterion relative to A and Y if all backdoor paths between A and Y are blocked given X and X does not include descendants of A

X satisfies backdoor criterion $\Leftrightarrow$ $Y(a) \perp\!\!\!\perp A|X$

Backdoor between A and Y: a path that starts with $A \leftarrow \cdots$

Front door criterion: identification rule 2 using DAG

- Can we identify $\mathbb{E}[Y(a)]$ in the DAG below?



Yes! Front door criterion.

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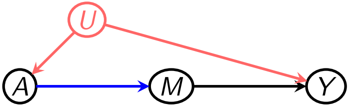

$p(M(a)) = p(M|A = a)$

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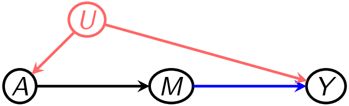

$p(Y(m)) = \int_a p(Y|M = m, A = a)p(a)\mathrm{d}a$

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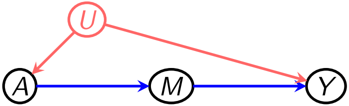

$\mathbb{E}[Y(a)] = \int_y \int_m y p(Y(m) = y) p(M(a) = m) \mathrm{d}m \mathrm{d}y$ but can you see why?

SWIGs: putting counterfactuals on DAGs

Recall the three identification conditions

1. Consistency: $Y = \sum_a Y(a)\mathbb{1}\{A = a\}$
2. Positivity/Overlap: $0 < \Pr(A = a|X) < 1 \ \forall a$
3. Ignorability/Randomization: $Y(a) \perp\!\!\!\perp A|X \ \forall a$

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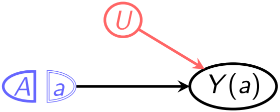
SWIG for RCT: splitting treatment node, then all descendants become counterfactuals



Reading independence: it is immediate $Y(a) \perp\!\!\!\perp A$ because A and $Y(a)$ are d-separated

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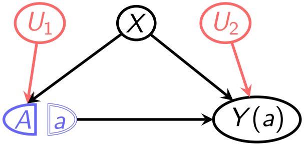
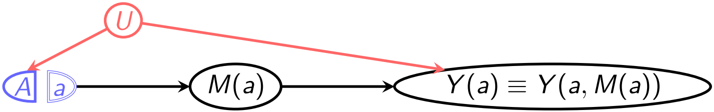


Reading conditional independence (CI): it is immediate $Y(a) \perp\!\!\!\perp A|X$ because $Y(a) \perp_{\mathcal{G}_{swig}} A|X$

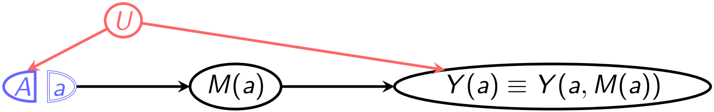
Front door with SWIG



so we have $(\star)$ $M(a) \perp\!\!\!\perp A$

$$\mathbb{E}[Y(a)] = \mathbb{E}[\mathbb{E}[Y(a)|M(a)]] = \int_m \mathbb{E}[Y(a)|M(a) = m]f(M(a) = m)\mathrm{d}m$$

Front door with SWIG



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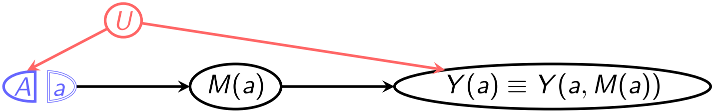

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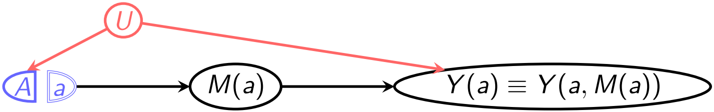

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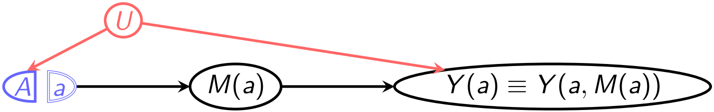

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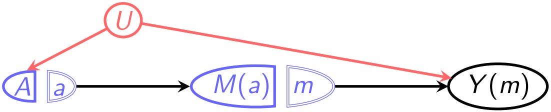
now we need a SWIG by intervening both A and M

Front door with SWIG



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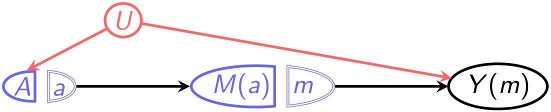

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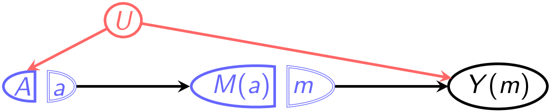
Front door with SWIG



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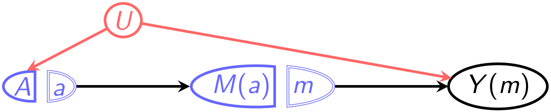

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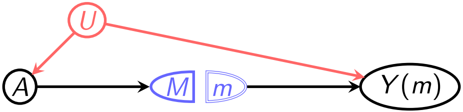
now we need a world by intervening M alone

Front door with SWIG



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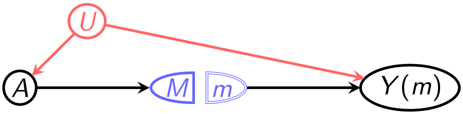
Front door with SWIG



so we have (⋆) $Y(m) \perp\!\!\!\perp M|A$

$$
\begin{aligned}
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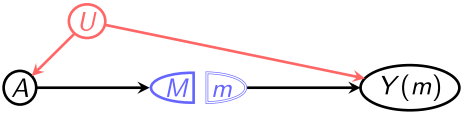
&\overset{(\star)}{=} \int_m \mathbb{E}_A[\mathbb{E}_{Y(m)}[Y(m)|A, M=m]] f(M=m|A=a)\mathrm{d}m \\
&= \int_m \mathbb{E}_A[\mathbb{E}_Y[Y|A, M=m]] f(M=m|A=a)\mathrm{d}m
\end{aligned}
$$

Front door with SWIG



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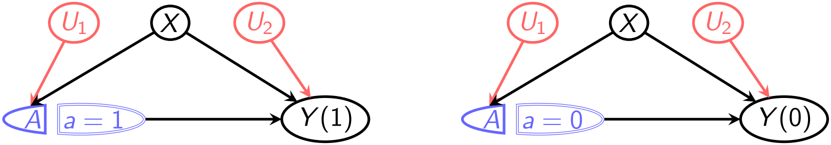

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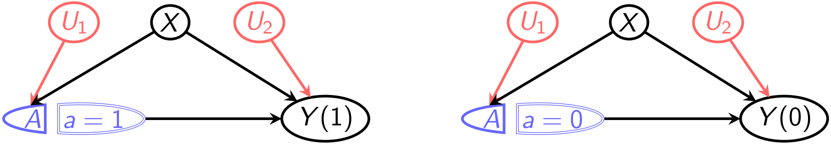

- $Y(1)$ and $Y(0)$ are never on the same graph! So the theory of SWIG cannot be used to identify the joint distribution of $(Y(1), Y(0))$

No cross-world identification – feature or bug?

- What do you think? Not allowing cross-world identification, is it an advantage or a disadvantage?

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- What do you think? Not allowing cross-world identification, is it an advantage or a disadvantage?
- Discuss among yourselves
- Pros: For causal inference in reality, we can never observe $Y(1)$ and $Y(0)$ for one person so there is no way to estimate their joint distribution without making assumptions

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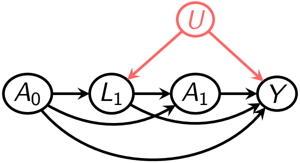
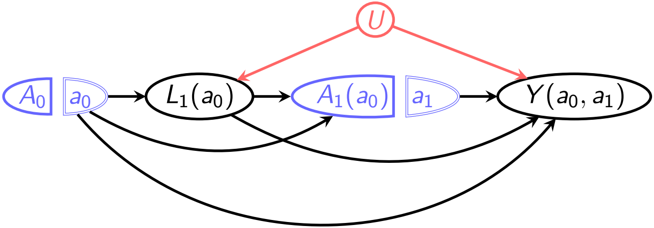
- What do you think? Not allowing cross-world identification, is it an advantage or a disadvantage?
- Discuss among yourselves
- Pros: For causal inference in reality, we can never observe $Y(1)$ and $Y(0)$ for one person so there is no way to estimate their joint distribution without making assumptions
- Cons: For those taking causality as a much higher-level meta-physical type of human inquiry, this is definitely limiting our "imagination power"

One more example of SWIG: Verma

Can we identify $\mathbb{E}[Y(a_0, a_1)]$ in the following complete Verma's graph?

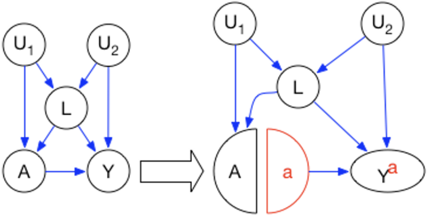


SWIG?



Reading CI: $Y(a_0, a_1) \perp\!\!\!\perp A_0$ and $Y(a_0, a_1) \perp\!\!\!\perp A_1(a_0) | L_1(a_0), A_0$. We will come back to this on Friday.

More Examples

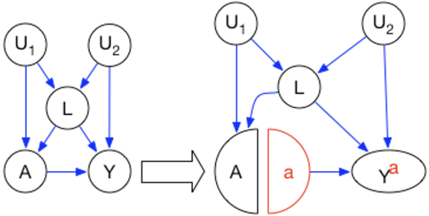


- We can read directly from the template that:

  $A \not\perp\!\!\!\perp Y^{a} \mid L.$

  – Conditioning on L opens the path $A \leftarrow U_1 \rightarrow L \leftarrow U_2 \rightarrow Y^{a}$.

More Examples



- We can also directly read that: $A \not\perp\!\!\!\perp Y^a$.
  - $A \leftarrow U_1 \rightarrow L \rightarrow Y^a$ and $A \leftarrow L \leftarrow U_2 \rightarrow Y^a$ are both open.
- So we have intractable confounding.

Non-Homework :-)

Read the following papers:
Richardson & Robins, Single world intervention graphs (148 pages);

Shpitser, Richardson & Robins, Multivariate Counterfactual Systems And Causal Graphical Models (34 pages)

Sensitivity Analysis

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Sensitivity analysis is a technique very commonly used in applied math, computational physics, or any area mathematical modeling is used: check how our results change by perturbing some parameters in the model; it is important for science

Have you done any sensitivity analysis yourself in your research?

Basic idea of sensitivity analysis

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Doll and Hill (1950 BMJ): smoking-lung cancer RR under ignorability $RR^{obs}_{AY} = \frac{\Pr(Y=1|A=1)}{\Pr(Y=1|A=0)} \approx 9$

Figure: Sir Austin Bradford Hill (1897-1991)

Fisher's dispute

Sir R.A. Fisher (founding father of statistics, also Donald B. Rubin's academic grand father) disagreed with Doll and Hill (Fisher 1957 BMJ)

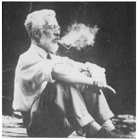

Figure: Sir Ronald Fisher (1890-1962)

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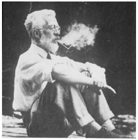

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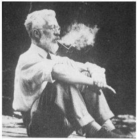

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Can you draw Fisher's DAG?

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Fisher is likely to be incorrect

How to get $RR_{AU} \geq RR_{AY}$?

Ignorability assumption ($\star$): $Y(a) \perp\!\!\!\perp A|U$, U unmeasured

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Observed RR:

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Causal RR:

$$RR_{AY}^{true} = \frac{\Pr(Y(1)=1)}{\Pr(Y(0)=1)} = \frac{\sum_{u=0,1}\Pr(Y(1)=1|U=u)\Pr(U=u)}{\sum_{u=0,1}\Pr(Y(0)=1|U=u)\Pr(U=u)}$$

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$$\overset{(\star)}{=} \frac{\sum_{u=0,1}\Pr(Y=1|A=1,U=u)\Pr(U=u)}{\sum_{u=0,1}\Pr(Y=1|A=0,U=u)\Pr(U=u)}$$

Sensitivity analysis: can we bound RR_{AY}^{true} by RR_{AY}^{obs}?

Two definitions: $A - U$ association and $U - Y(a)$ association by RR

$$RR_{AU} := \frac{\Pr(U=1|A=1)}{\Pr(U=1|A=0)}$$

$$RR_{UY(a)} := \max\left\{\frac{\Pr(Y(a)=1|U=1)}{\Pr(Y(a)=1|U=0)}, \frac{\Pr(Y(a)=1|U=0)}{\Pr(Y(a)=1|U=1)}\right\}$$

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BOUND (REF: Ding, VanderWeele. Sensitivity analysis without assumptions. Epidemiology (2017)): if $RR_{AY}^{obs} > 1$

$$\frac{RR_{AY}^{obs}}{RR_{AY}^{true}} \leq \frac{RR_{AU}RR_{UY}}{RR_{AU} + RR_{UY} - 1}$$

Proof sketch

Recall

$$RR_{AY}^{true} = \frac{\sum_{u=0,1} \Pr(Y=1|A=1,U=u)\Pr(U=u)}{\sum_{u=0,1} \Pr(Y=1|A=0,U=u)\Pr(U=u)}$$

Introduce two new notation

$$RR_{AY}^{true,+} = \frac{\sum_{u=0,1} \Pr(Y=1|A=1,U=u)\Pr(U=u|A=1)}{\sum_{u=0,1} \Pr(Y=1|A=0,U=u)\Pr(U=u|A=1)}$$

$$RR_{AY}^{true,-} = \frac{\sum_{u=0,1} \Pr(Y=1|A=1,U=u)\Pr(U=u|A=0)}{\sum_{u=0,1} \Pr(Y=1|A=0,U=u)\Pr(U=u|A=0)}$$

With some algebra, one can show

$$RR_{AY}^{true} = wRR_{AY}^{true,+} + (1-w)RR_{AY}^{true,-}$$

with

$$w = \frac{\sum_{u=0,1} \Pr(Y=1|A=0,U=u)\Pr(U=u,A=1)}{\sum_{u=0,1} \Pr(Y=1|A=0,U=u)\Pr(U=u,A=1)+\sum_{u=0,1} \Pr(Y=1|A=0,U=u)\Pr(U=u,A=0)}$$

Proof sketch

Similarly,

$$\left(\frac{RR_{AY}^{obs}}{RR_{AY}^{true}}\right)^{-1} = w\left(\frac{RR_{AY}^{obs}}{RR_{AY}^{true,+}}\right)^{-1} + (1-w)\left(\frac{RR_{AY}^{obs}}{RR_{AY}^{true,-}}\right)^{-1}$$

Now

$$\frac{RR_{AY}^{obs}}{RR_{AY}^{true,+}} = \frac{\frac{\sum_{u=0,1}\Pr(Y=1|A=1,U=u)\Pr(U=u|A=1)}{\sum_{u=0,1}\Pr(Y=1|A=0,U=u)\Pr(U=u|A=0)}}{\frac{\sum_{u=0,1}\Pr(Y=1|A=1,U=u)\Pr(U=u|A=1)}{\sum_{u=0,1}\Pr(Y=1|A=0,U=u)\Pr(U=u|A=1)}}$$

$$= \frac{\sum_{u=0,1}\Pr(Y=1|A=0,U=u)\Pr(U=u|A=1)}{\sum_{u=0,1}\Pr(Y=1|A=0,U=u)\Pr(U=u|A=0)}$$

$$= \frac{RR_{AU}\Pr(U=1|A=0)\left(\Pr(Y=1|A=0,U=1)-\Pr(Y=1|A=0,U=0)\right)+\Pr(Y=1|A=0,U=0)}{\Pr(U=1|A=0)\left(\Pr(Y=1|A=0,U=1)-\Pr(Y=1|A=0,U=0)\right)+\Pr(Y=1|A=0,U=0)}$$

$$= \frac{RR_{AU}\Pr(U=1|A=0)\left(RR_{UY(0)}-1\right)+1}{\Pr(U=1|A=0)\left(RR_{UY(0)}-1\right)+1}$$

$$\leq \frac{RR_{AU}RR_{UY(0)}}{RR_{AU}+RR_{UY(0)}-1}$$

in the last step, maximized at $RR_{AU}\Pr(U=1|A=0)=1$.

How to use such a bound?

$$\underbrace{\frac{RR_{AU}RR_{UY}}{RR_{AU} + RR_{UY} - 1}}_{\text{Bounding Factor (BF)}} \geq \frac{RR_{AY}^{obs}}{RR_{AY}^{true}}$$

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To reduce the observed RR to the hypothetical true causal RR (e.g. no effect, taking value 1), BF needs to exceed RR_{AY}^{obs} to explain away the observed RR

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Connect to Cornfield's analysis?

Fact 1

$$a \geq \frac{ab}{a+b-1} \text{ if } a \geq 1$$

so $RR_{AU} \geq BF$ and $RR_{UY} \geq BF$

BF table

bounding factor		RR_{UD} 1.3	1.5	1.8	2	2.5	3	3.5	4	5
	1.3	1.06	1.08	1.11	1.13	1.16	1.18	1.20	1.21	1.23
	1.5	1.08	1.12	1.17	1.20	1.25	1.29	1.31	1.33	1.36
	1.8	1.11	1.17	1.25	1.29	1.36	1.42	1.47	1.50	1.55
	2	1.13	1.20	1.29	1.33	1.43	1.50	1.56	1.60	1.67
RR_{EU}	2.5	1.16	1.25	1.36	1.43	1.56	1.67	1.75	1.82	1.92
	3	1.18	1.29	1.42	1.50	1.67	1.80	1.91	2.00	2.14
	3.5	1.20	1.31	1.47	1.56	1.75	1.91	2.04	2.15	2.33
	4	1.21	1.33	1.50	1.60	1.82	2.00	2.15	2.29	2.50
	5	1.23	1.36	1.55	1.67	1.92	2.14	2.33	2.50	2.78

E-value

VanderWeele, Ding. Sensitivity Analysis in Observational Research: Introducing the E-Value. Annals of Internal Medicine (2017)
!!Also read the response letter and the authors' rejoinder!!

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To explain away the observed RR, we need $\frac{\rho^2}{2\rho-1} \geq RR_{AY}^{obs}$, which gives us the quadratic inequality: when $RR_{AY}^{obs} \geq 1$

$$\rho^2 - 2RR_{AY}^{obs}\rho + RR_{AY}^{obs} \geq 0$$
$$\Rightarrow \rho \geq RR_{AY}^{obs} + \sqrt{RR_{AY}^{obs}(RR_{AY}^{obs} - 1)}$$

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What if $RR_{AY}^{obs} \leq 1$?

Other sensitivity analysis strategies

Most other sensitivity analysis strategies rely on further untestable assumptions to argue against the untestable ignorability assumption

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For example, one could postulate the following "exponential tilting model"

$$\frac{p(Y(a) = y|X, A = 1 - a)}{p(Y(a) = y|X, A = a)} = \frac{\exp\{\gamma_a f_a(y)\}}{\mathbb{E}\left[\exp\{\gamma_a f_a(Y)\}|X, A = a\right]}$$

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With such a model, one immediately have

$$\mathbb{E}[Y(a)] = \int_x \left\{ \begin{array}{c} \mathbb{E}[Y|X = x, A = a]\Pr(A = a|X = x) \\ + \frac{\mathbb{E}[Y \exp\{\gamma_a f_a(Y)\}|X=x, A=a]}{\mathbb{E}[\exp\{\gamma_a f_a(Y)\}|X=x, A=a]}\Pr(A = 1 - a|X = x) \end{array} \right\} p(x)\mathrm{d}x$$

and hence the ATE can be identified as

$$\tau(\gamma_0, \gamma_1; f_0, f_1) = \mathbb{E}[Y(1)] - \mathbb{E}[Y(0)]$$

Floor discussion

Why do you think one postulate the sensitivity analysis model as

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"Counterfactuals are the ultimate unmeasured confounder"
– REF: Robins et al. Sensitivity analyses for unmeasured confounding assuming a marginal structural model for repeated measures. Stats in Med (2004).

In summary

In general there are two different strategies to perform sensitivity analysis: one relatively more straightforward, one calling for deeper theoretical analysis

- Given estimated causal effect, obtain the strength of unmeasured confounding to explain away the effect (e.g. E-value type analysis, becoming standard in medical practice and getting popular in industry)

In summary

In general there are two different strategies to perform sensitivity analysis: one relatively more straightforward, one calling for deeper theoretical analysis

- Given estimated causal effect, obtain the strength of unmeasured confounding to explain away the effect (e.g. E-value type analysis, becoming standard in medical practice and getting popular in industry)

- Postulating a model that incorporates unmeasured confounding, then estimate the causal effect using the most advanced statistical methodology and see how the result changes with the sensitivity parameter γ
  If interested in further theory, take a look at:
  REF: Scharfstein et al. Semiparametric Sensitivity Analysis: Unmeasured Confounding in Observational Studies. 2021
  and see how they developed the theoretical results for sensitivity analysis


self-esteem
grades
happiness

Mediation analysis

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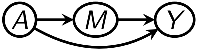

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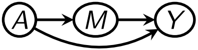

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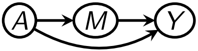

- Mechanistic question: does A directly cause Y or A causes Y through M or both?
- Can you create a story based on this graph?

Examples

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- Machine learning fairness: A gender, Y college admission, M applying to department with lower admission rate

mediation analysis: motivations

Mediation questions:

- You have a theory for why the effect of a treatment/exposure on the outcome is mediated by > 1 variables
- You wish to frame your study in terms of causal questions, including hypothetical interventions

Non-mediation questions:

- Is it better to intervene on the treatment or the mediator (if you cannot do both)?
- What are the various effects of treatment?

How would you proceed?

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- Jamie Robins in his 1986 paper (g-formula) has defined the concept direct and indirect effect
- Around the same time, Judea Pearl also started to consider direct and indirect effect

Zoo of direct and indirect effects

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Zoo of direct and indirect effects

- Treatment $A \in \{0, 1\}$, Mediator $M \in \{0, 1\}$, Outcome Y

- Total effect of A on Y:

$$\tau_{tot} := \mathbb{E}[Y(1) - Y(0)] \equiv \mathbb{E}[Y(1, M(1)) - Y(0, M(0))]$$

- Controlled direct effect (CDE) of A on Y:

$$\tau_{cde}(0) := \mathbb{E}[Y(a = 1, m = 0) - Y(a = 0, m = 0)]$$
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Zoo of direct and indirect effects

- Treatment $A \in \{0, 1\}$, Mediator $M \in \{0, 1\}$, Outcome Y
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$$\tau_{cie}(0) := \mathbb{E}[Y(a = 0, m = 1) - Y(a = 0, m = 0)]$$
$$\tau_{cie}(1) := \mathbb{E}[Y(a = 1, m = 1) - Y(a = 1, m = 0)]$$

Warning about CDE

- CDE is NOT for mediation analysis, as we have discussed

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- Natural direct effect (NDE) of A on Y: the part of τ that does not go through M

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- Natural indirect effect (NIE) of A on Y: the part of τ that does not go through M

$$\tau_{nie}(0) := \mathbb{E}[Y(0, M(1)) - Y(0, M(0))]$$
$$\tau_{nie}(1) := \mathbb{E}[Y(1, M(1)) - Y(1, M(0))]$$

Trivial decompositions

$$\tau_{tot} = \mathbb{E}[Y(1, M(1)) - Y(0, M(0))] = \tau_{nde}(1) + \tau_{nie}(0)$$
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Not so trivial 4-way decomposition: connections with causal interactions (study on your own)

A less trivial but useful decomposition of the total effect τ_{tot}
REF: VanderWeele, A Unification of Mediation and Interaction: A 4-Way Decomposition. Epidemiology (2014)

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A less trivial but useful decomposition of the total effect τ_{tot}

REF: VanderWeele, A Unification of Mediation and Interaction: A 4-Way Decomposition. Epidemiology (2014)

For simplicity, assume all variables are $\{0, 1\}$-valued

$$
\begin{aligned}
\tau_{tot} &= \mathbb{E}[Y(1, M(1)) - Y(0, M(1))] + \tau_{nie}(0) \\
&= \mathbb{E}[Y(1, M(0)) - Y(0, M(0))] + \mathbb{E}[Y(1, M(1)) - Y(0, M(1)) - Y(1, M(0)) + Y(0, M(0))] + \tau_{nie}(0) \\
&= \mathbb{E}[Y(1, 0) - Y(0, 0)] + \mathbb{E}[Y(1, M(0)) - Y(0, M(0)) - Y(1, 0) + Y(0, 0)] \\
&\quad + \mathbb{E}[Y(1, M(1)) - Y(0, M(1)) - Y(1, M(0)) + Y(0, M(0))] + \tau_{nie}(0) \\
&= \tau_{cde}(0) + \mathbb{E}[Y(1, M(0)) - Y(0, M(0)) - Y(1, 0) + Y(0, 0)] \\
&\quad + \mathbb{E}[Y(1, M(1)) - Y(0, M(1)) - Y(1, M(0)) + Y(0, M(0))] + \tau_{nie}(0).
\end{aligned}
$$

$$
\begin{aligned}
& \mathbb{E}[Y(1, M(0)) - Y(0, M(0)) - Y(1,0) + Y(0,0)] \\
= \; & \mathbb{E}\left[\sum_{m=0,1} \{Y(1,m) - Y(0,m)\} \mathbb{1}\{M(0) = m\} - Y(1,0) + Y(0,0)\right] \\
= \; & \mathbb{E}\left[\left\{\sum_{m=0,1} Y(1,m) - Y(0,m) - Y(1,0) + Y(0,0)\right\} \mathbb{1}\{M(0) = 1\}\right] \\
= \; & \underbrace{\mathbb{E}\left[\underbrace{\{Y(1,1) - Y(0,1) - Y(1,0) + Y(0,0)\}}_{\text{interaction between } a \text{ and } m} \mathbb{1}\{M(0) = 1\}\right]}_{\text{reference interaction}}
\end{aligned}
$$

$$
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&\mathbb{E}[Y(1, M(1)) - Y(0, M(1)) - Y(1, M(0)) + Y(0, M(0))] \\
&= \mathbb{E}\left[\sum_{m=0,1} \{Y(1, m) - Y(0, m)\}\{\mathbb{1}\{M(1) = m\} - \mathbb{1}\{M(0) = m\}\}\right] \\
&= \underbrace{\mathbb{E}\left[\underbrace{\{Y(1,1) - Y(0,1) - Y(1,0) + Y(0,0)\}}_{\text{interaction between } a \text{ and } m}\{\mathbb{1}\{M(1) = 1\} - \mathbb{1}\{M(0) = 1\}\}\right]}_{\text{mediated interaction}}
\end{aligned}
$$

To summarize:

$$
\tau_{tot} = \tau_{cde}(0) + \text{reference interaction} + \text{mediated interaction} + \tau_{nie}(0)
$$

How to identify natural direct/indirect effect?

Consider NDE: $\tau_{nde}(0) := \mathbb{E}[Y(1, M(0)) - Y(0, M(0))]$

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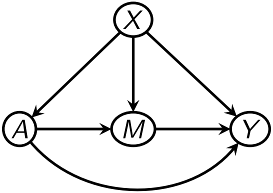
Discuss why

How to identify $\mathbb{E}[Y(1, M(0))]$?

Mediation DAG with confounders



REF: Pearl, Direct and indirect effects, UAI (2001) consider the following ignorability conditions:

1. $Y(a,m) \perp\!\!\!\perp A|X$: no unmeasured treatment-outcome confounder
2. $Y(a,m) \perp\!\!\!\perp M|\{X,A\}$: no unmeasured mediator-outcome confounder
3. $M(a) \perp\!\!\!\perp A|X$: no unmeasured treatment-mediator confounder
4. $Y(a,m) \perp\!\!\!\perp M(a')|X$: will come back later

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$$
\begin{aligned}
&\mathbb{E}[Y(1, M(0))] \\
&= \mathbb{E}_X\left[\mathbb{E}_{Y(1,M(0))}[Y(1, M(0))|X, M(0)]\right] \\
&= \int_x \int_m \left\{ \int_y y f(Y(1, M(0)) = y|X = x, M(0) = m)\mathrm{d}y \right\} f(X = x, M(0) = m)\mathrm{d}m\mathrm{d}x \\
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&\overset{1}{=} \int_x \int_m \left\{ \int_y y f(Y(1, m) = y|X = x, A = 1)\mathrm{d}y \right\} f(M = m|X = x, A = 0) f(X = x)\mathrm{d}m\mathrm{d}x \\
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&= \int_x \int_m \mathbb{E}[Y|X = x, A = 1, M = m] f(M = m|X = x, A = 0) f(X = x)\mathrm{d}m\mathrm{d}x
\end{aligned}
$$

SWIG for mediation DAG

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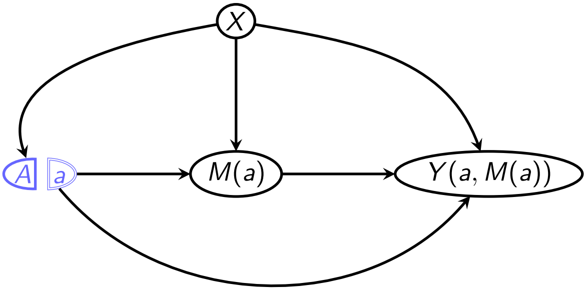

Issue: SWIG does not allow $Y(a, M(a'))$ on the graph for $a \neq a'$ because it is a cross-world counterfactual

Philosophical forkpath

Based on the causal model defined via SWIG, impossible to identify NDE or NIE because they are cross-world counterfactuals

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Pearl's causal model is called "Non-Parametric Structural Equation Models with Independent Errors (NPSEM-IE)"

Therefore NPSEM $\subset$ FFRCISTG

Abstract difference between FFRCISTG vs. NPSEM-IE

Let's look at the key difference between FFRCISTG vs. NPSEM-IE

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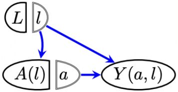
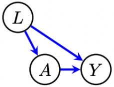
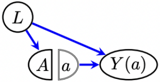
FFRCISTG/SWIG

	One-Step Ahead Counterfactuals		Passive Observation		Experimental Intervention on A

Graph:







	One-Step Ahead Counterfactuals		Passive Observation		Experimental Intervention on A
Variables:	L $A(l)$ $Y(a,l)$	$\Rightarrow$	L $A \equiv A(L)$ $Y \equiv Y(A,L)$	$\Rightarrow$	L $A \equiv A(L)$ $Y(a) \equiv Y(a,L)$
Interpretation:	Counterfactuals when A and L are intervened on;		Observed System;		Prior to intervention: L and A; after intervention: $Y(a)$.
Meaning of A:	(A does not appear)		Natural value of A;		Natural value of A (observed prior to intervention on A).

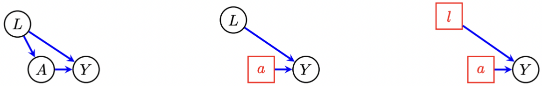
Single World No Confounding Assumption: for each pair a, l: $L \perp\!\!\!\perp A(l) \perp\!\!\!\perp Y(a,l)$

Abstract difference between FFRCISTG vs. NPSEM-IE

Let's look at the key difference between FFRCISTG vs. NPSEM-IE

NPSEM-IE

	Passive Observation		Experimental Intervention on A		Experimental Intervention on A and L
Graph:					



	Passive Observation		Experimental Intervention on A		Experimental Intervention on A and L
Variables:	$L = f_L(\varepsilon_L)$ $A = f_A(L, \varepsilon_A)$ $Y = f_Y(A, L, \varepsilon_Y)$	$\Rightarrow$	$L = f_L(\varepsilon_L)$ $A = a$ $Y = f_Y(A, L, \varepsilon_Y)$	$\Rightarrow$	$L = l$ $A = a$ $Y = f_Y(A, L, \varepsilon_Y)$
Interpretation:	Observed system;		Variables in system in which A is set to a;		L and A after each is intervened on; Y after both interventions.
Meaning of A:	Natural value of A;		Value of A after intervention on A;		Value of A after intervention on A (and L).

Independent Errors No Confounding Assumption: $\varepsilon_L \perp\!\!\!\perp \varepsilon_A \perp\!\!\!\perp \varepsilon_Y$

Relationships to Counterfactuals:

Error terms:	$\varepsilon_L = L$;	$\varepsilon_A = \{A(l) \text{ for all } l\}$;	$\varepsilon_Y = \{Y(a, l) \text{ for all } l, a\}$
Structural equations:	$L = f_L(\varepsilon_L)$	$A(l) = f_A(l, \varepsilon_A)$	$Y(a, l) = f_Y(a, l, \varepsilon_Y)$

Difference presented in math

- Pearl's NPSEM-IE: for all variables $V_1, \cdots, V_N$ on a causal DAG $\mathcal{G}$:

$$V_1 = f_1(\text{pa}_1; \varepsilon_1)$$
$$\vdots$$
$$V_N = f_N(\text{pa}_N; \varepsilon_N)$$

s.t. $\{V_1\} \perp\!\!\!\perp \{V_2(x_{\text{pa}_2}); \forall x_{\text{pa}_2}\} \perp\!\!\!\perp \cdots \perp\!\!\!\perp \{V_N(x_{\text{pa}_N}); \forall x_{\text{pa}_N}\}$

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· Robins' FFRCISTG/SWIG: for all variables $V_1, \cdots, V_N$ on a causal DAG $\mathcal{G}$:
for each x_V: $V_1 \perp\!\!\!\perp V_2(x_{\text{pa}_2}) \perp\!\!\!\perp \cdots \perp\!\!\!\perp V_N(x_{\text{pa}_N})$

What to do without cross-world independence assumption?

When one cannot make further progress with the current definition, then change the definition

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When one cannot make further progress with the current definition, then change the definition

Here comes "Interventional Direct/Indirect Effect (IDE/IIE)":
REF: VanderWeele, Vansteelandt, Robins. Effect decomposition in the presence of an exposure-induced mediator-outcome confounder. Epidemiology (2014).

Definition of IDE/IIE

How do we change the definition(s)? Let's look at the derivation of $\mathbb{E}[Y(1, M(0))]$, where we use the cross-world independence condition (4)

$$
\begin{aligned}
&\mathbb{E}[Y(1, M(0))] \\
&= \mathbb{E}_X\left[\mathbb{E}_{Y(1,M(0))}[Y(1, M(0))|X, M(0)]\right] \\
&= \int_x \int_m \left\{ \int_y y f(Y(1, M(0)) = y|X = x, M(0) = m)\mathrm{d}y \right\} f(X = x, M(0) = m)\mathrm{d}m\mathrm{d}x \\
&= \int_x \int_m \left\{ \int_y y f(Y(1, m) = y|X = x, M(0) = m)\mathrm{d}y \right\} f(M(0) = m|X = x) f(X = x)\mathrm{d}m\mathrm{d}x \\
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\end{aligned}
$$

Definition of IDE/IIE

Cross-world assumption is used, from $\mathbb{E}[Y(1, M(0))]$ to

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Can you give a meaning to this quantity?

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Can you give a meaning to this quantity?

$$\mathbb{E}\left[Y(1, \widetilde{M}_{0|X}) \right]$$

where $\widetilde{M}_{0|X}$ is a random draw from the probability distribution of $M(0)|X$ independent of everything else

Definition of IDE/IIE

So we eventually define interventional direct and indirect effects as:

$$\tau_{ide}(0) \coloneqq \mathbb{E}\left[Y(1, \widetilde{M}_{0|X})\right] - \mathbb{E}\left[Y(0, \widetilde{M}_{0|X})\right]$$
$$\tau_{ide}(1) \coloneqq \mathbb{E}\left[Y(1, \widetilde{M}_{1|X})\right] - \mathbb{E}\left[Y(0, \widetilde{M}_{1|X})\right]$$
$$\tau_{iie}(0) \coloneqq \mathbb{E}\left[Y(0, \widetilde{M}_{1|X})\right] - \mathbb{E}\left[Y(0, \widetilde{M}_{0|X})\right]$$
$$\tau_{iie}(1) \coloneqq \mathbb{E}\left[Y(1, \widetilde{M}_{1|X})\right] - \mathbb{E}\left[Y(1, \widetilde{M}_{0|X})\right]$$

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$$\tau_{iie}(0) \coloneqq \mathbb{E}\left[Y(0, \widetilde{M}_{1|X})\right] - \mathbb{E}\left[Y(0, \widetilde{M}_{0|X})\right]$$
$$\tau_{iie}(1) \coloneqq \mathbb{E}\left[Y(1, \widetilde{M}_{1|X})\right] - \mathbb{E}\left[Y(1, \widetilde{M}_{0|X})\right]$$

What do you think of this level of rigor when it comes to impose scientific meaning to a parameter?

Other related definitions of direct/indirect effects

Other than IDE/IIE, people have developed other definitions of direct/indirect effects, but they all have similar spirit to IDE/IIE

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Organic direct/indirect effect
REF: Lok, Bosch. Causal organic indirect and direct effects: closer to Baron and Kenny. Epidemiology (2021).

Mediators are often difficult to even imagine an intervention (e.g. BMI) so $Y(a, m)$ is ill-defined
Organic DE/IE: hypothesize an organic intervention I on mediator that does not have a direct effect on Y but $M(0, I = 1)|X \sim M(1)|X$. Then, for $a = 0, 1$

$$\tau_{ode}(a) := \mathbb{E}[Y(1, I = a) - Y(0, I = a)]$$
$$\tau_{oie}(a) := \mathbb{E}[Y(a, I = 1) - Y(a, I = 0)]$$

Interpretation of ODE/OIE

Example? A blood pressure drug, M blood pressure, Y heart attack
What is I?

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I: Reduction in salt intake in diet; salt should only cause Y through M, so no direct effect on Y

Others: learn on your own

Separable effects:
REF: Robins, Richardson, Shpitser. An Interventionist Approach to Mediation Analysis.
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Population direct/indirect effect:
REF: Fulcher, Shpitser, Marealle, Tchetgen Tchetgen. Robust inference on population indirect causal effects: The generalized front-door criterion.

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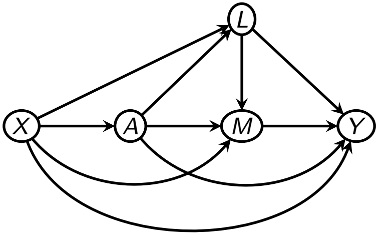

Total effect τ_{tot} of A on Y can be decomposed into

$$\tau_{tot} = \tau_{A\to Y} + \tau_{A\to M\to Y} + \tau_{A\to L\to Y} + \tau_{A\to L\to M\to Y}$$

Next time

Time-varying causal inference, (Optimal) dynamic treatment regimes, dynamic regime SWIG

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"Soft intervention" loosely means stochastic treatment regimes, e.g.
$A \sim \mathrm{Bernoulli}(\text{softmax (blood pressure)})$

Any Questions?